ECE 341: Probability and Random Processes for Engineers, Spring 2012
Homework 7
Name:
Assigned: 02.22.2012
Due: 02.29.2012

Problem 1. Given a uniform, continuous random variable X whose range is $[-3, 3]$. We quantize X to give Y, using L levels such that $\text{SNR}_Q(\text{dB}) = 25$. Find:

1. $E[X^2]$
2. $Var[X]$
3. μ_X
4. $E[(X-Y)^2]$
5. L
6. Verify using Matlab that your choice of L yields $\text{SNR}_Q(\text{dB}) \approx 25$

Solution 1:
The PDF of X is a continuous Random Variable then

$$f_X(x) = \begin{cases} \frac{1}{6} & -3 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

The Variance for a Uniform Random Variable is as follows

$$Var[X] = \frac{(b-a)^2}{12} \quad (1)$$

1. $E(X^2)$ is calculated as follows

$$E(X^2) = \int_{-3}^{3} f_X(x)x^2 dx = \int_{-3}^{3} \frac{1}{6}x^2 dx = 3 \quad (2)$$

2. $Var[X]$ is calculated as follows

$$Var[X] = \frac{(b-a)^2}{12} = \frac{(3-(-3))^2}{12} = 3 \quad (3)$$

3. μ_X is calculated as follows

$$\mu_X = \sqrt{E(X^2) - Var(X)} = \sqrt{3-3} = 0 \quad (4)$$

4.Number of levels L is calculated as follows

$$10\log SNR = 25 => SNR = 10^{2.5} = 316.23 \quad (5)$$
$$L^2 = 316.23 => L = 17.778 \quad (6)$$

But L can only be integer representing number of levels for quantization thus choose $L = 18$.

5. $E(X-Y)^2$ known as the mean square error. Where (X-Y) represent the error due to quantizations is calculated as follows

$$\Delta = \frac{b-a}{L} = \frac{6}{18} = \frac{1}{3} \quad (7)$$
$$(8)$$
$$E(X-Y)^2 = \frac{\Delta^2}{12} = \frac{\frac{1}{3}^2}{12} = 0.00926. \quad (9)$$

6. MATLAB code is as follows:

```
>> X = rand(1,1e7)*6-3;
>> delta = 6/L;
>> Y = round(X/delta)*delta;
>> Z = X-Y;
>> mean(Z.^2)

ans =

   0.00925335563324

>> 10*log10(mean(X.^2)/mean(Z.^2))

ans =

  25.10618609681968
```

Problem 2. Let W =Bernoulii(1/2) and $X = 10W - 5$ and $Y = X + N$, where N is a Gaussian random variable having zero mean. Define the SNR as $E[X^2]/E[N^2]$, or in decibels, SNR(dB)=$10\log_{10}(E[X^2]/E[N^2])$. Define the decoder output (our decision on which bit was transmitted based on the received signal Y) as a new random variable Z which is equal to 0 is $Y < 0$ and equal to 1 is $Y \geq 0$. If $Z = W$ then no error occurs, if $Z \neq W$ then an error occurred due to the additive Gaussian noise in the channel. Find:

1. The variance of N when the channel SNR is 30dB.

2. The channel SNR in dB when the variance of the RV N is 0.1.

3. The power of the noise in the channel is $E[N^2]$. Derive an expression for the probability of a bit error in the channel (probability that $Z \neq W$) in terms of the noise power.

4. Derive an expression for the probability of a bit error in the channel in terms of channel SNR.

Solution 2: W= Bernoulli (1/2) . Thus W takes on values either 1 or 0 with equal probability $p = 0.5$ which makes the range of values for $X = 5$ or -5 since $X = 10W - 5$. It is important for this problem to calculate the $E(X^2)$ which is

$$E(X^2) = (5)^2(0.5) + (-5)^2(0.5) = 25 \tag{10}$$

1. The variance of N can be calculated from the SNR.

$$10 \log SNR = 10 \log \frac{E(X^2)}{E(N^2)} = 30 \tag{11}$$

$$\frac{E(X^2)}{E(N^2)} = 1000 \text{ with } E(X^2) = 25 \tag{12}$$

$$E(N^2) = \frac{25}{1000} = \frac{1}{40} \tag{13}$$

Note that $E(N^2) = \sigma_N^2$ because N is zero mean thus $E(N)$= 0.

2.SNR is calculated as follows

$$SNR(dB) = 10 \log \frac{E(X^2)}{\sigma_N^2} = 10 \log \frac{25}{0.1} = 23.9794 \tag{14}$$

$$\tag{15}$$

3. The Probability of a bit error in the channel (probability that $Z \; neqW$)

$$P(\text{bit error}) = P(Z \neq 1|W = 1)P(W = 1) + P(Z \neq 0|W = 0)P(W = 0) \tag{16}$$

$$= 0.5P(Y < 0|W = 1) + 0.5P(Y \geq 0|W = 0) \tag{17}$$

$$= 0.5P(X + N < 0|W = 1) + 0.5P(X + N \geq 0|W = 0) \tag{18}$$

$$= 0.5P(N < -5) + 0.5P(N \geq 5) \tag{19}$$

$$\text{But the the above probabilities are equal then} \tag{20}$$

$$= P(N \geq 5) = Q(\frac{5}{E[N^2]}) \tag{21}$$

4. The expression for the probability of a bit error in the channel in terms of channel SNR, from the above part it can be seen that:

$$P[\text{bit error}] = Q(\sqrt{\frac{E[X^2]}{E[N^2]}}) = Q(\sqrt{SNR}) \tag{22}$$

Problem 3. Let (X, Y) have the joint pmf given in the table below.

$Y = 3$	0.1	0.1	0
$Y = 2$	0	0.2	0.2
$Y = 1$	0	0.3	0.1
	$X = 1$	$X = 2$	$X = 3$

Find:

1. The pmf of X
2. The pmf of Y
3. $P[X = Y]$
4. $P[X > Y]$

Solution 3:
(a)The pmf of X is given by the column sums:

$$P_X(1) = 0.1, P_X(2) = 0.3 + 0.2 + 0.1 = 0.6, P_X(3) = 0.1 + 0.2 = 0.3. \tag{23}$$

(b)The pmf of Y is given by the rows sums:

$$P_Y(1) = 0.3 + 0.1 = 0.4, P_Y(2) = 0.2 + 0.2 = 0.4, P_Y(3) = 0.1 + 0.1 = 0.2. \tag{24}$$

(c) For $P[X = Y]$

$$P(X = Y) = p_{X,Y}(1, 1) + p_{X,Y}(2, 2) + p_{X,Y}(3, 3) = 0 + 0.2 + 0 = 0.2. \tag{25}$$

(d)For $P[X > Y]$

$$P(X > Y) = p_{X,Y}(2, 1) + p_{X,Y}(3, 1) + p_{X,Y}(3, 2) = 0.3 + 0.1 + 0.2 = 0.6. \tag{26}$$