ECE 341: Probability and Random Processes for Engineers, Spring 2012
Homework 6
Name:
Assigned: 02.15.2012
Due: 02.22.2012

Problem 1. A certain deep space transmitter uses on-off modulation of a laser to send a bit, with value either zero or one. If the bit is zero, the number of photons, X, arriving at the receiver has the Poisson distribution with mean $\lambda_0 = 2$; and if the bits is one, X has the Poisson distribution with mean $\lambda_1 = 6$. A decision rule is needed to decide, based on observation of X, whether the bit was a zero or a one.

1. Suppose we decide that a 1 is sent if $P[X = k|\text{one is sent}] > P[X = k|\text{zero is sent}]$. What does this decision rule simplify to in terms of X?
2. Suppose we know that the probability of sending a zero is π_0 and that of sending a one is π_1. Suppose now that we decide a 1 is sent if $P[X = k, \text{one is sent}] > P[X = k, \text{zero is sent}]$. What does this decision rule simplify to now in terms of X if $\pi_0 = 5\pi_1$?

Solution 1:

1. Decision Rule is in favor of 1 if

$$P[X = k|\text{one is sent}] \geq P[X = k|\text{zero is sent}] \tag{1}$$

$$\tag{2}$$

$$\text{Thus} \quad \frac{P[X = k|\text{one is sent}]}{P[X = k|\text{zero is sent}]} \geq 1 \tag{3}$$

But it is given that the number of photons X arriving at the receiver has Poisson distribution, then

$$\frac{P[X = k|\text{one is sent}]}{P[X = k|\text{zero is sent}]} = \frac{e^{-\lambda_1}\frac{\lambda_1^k}{k!}}{e^{-\lambda_0}\frac{\lambda_0^k}{k!}} = (\frac{\lambda_1}{\lambda_0})^k e^{-(\lambda_1-\lambda_0)} = 3^k e^{-4} \approx \frac{3^k}{54.6} \tag{4}$$

Thus based on this result, we decide that a bit 1 is sent if $\frac{3^k}{54.6} \geq 1$, that is if $X = k \geq 4$.

2. Now Decision Rule is in favor of 0 if

$$P[X = k, \text{one is sent}] \geq P[X = k, \text{zero is sent}] \tag{5}$$

$$\tag{6}$$

$$P[X = k|\text{one is sent}]P[\text{one is sent}] \geq P[X = k|\text{zero is sent}]P[\text{zero is sent}] \tag{7}$$

$$\tag{8}$$

$$P[X = k|\text{one is sent}]\pi_1 \geq P[X = k|\text{zero is sent}]\pi_0 \tag{9}$$

$$\tag{10}$$

$$\frac{P[X = k|\text{one is sent}]}{P[X = k|\text{zero is sent}]} \geq \frac{\pi_0}{\pi_1} = 5 \tag{11}$$

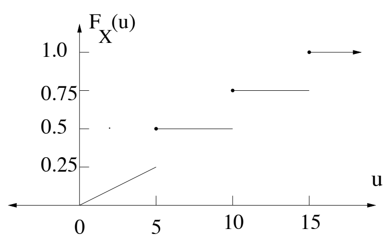
$$\tag{12}$$

$$\frac{P[X = k|\text{one is sent}]}{P[X = k|\text{zero is sent}]} = \frac{3^k}{54.6} \geq 5 \tag{13}$$

Thus the condition above is satisfied when $X \geq 6$

Problem 2. Let X have CDF shown in Fig. . Find the numerical values of the following quantities:

1. $P[X \leq 1]$
2. $P[X \leq 10]$
3. $P[X \geq 10]$
4. $P[X = 10]$
5. $P[|X - 5| \leq 0.1]$



Solution 2:

- $P[X \leq 1] = F_X[1] = 0.05$ where $F_X(1)$ is calculated from the equation of the line $F_X[x] = 0.05x$ when $x < 5$
- $P[X \leq 10] = F_X[10] = 0.75$ from the graph.
- $P[X \geq 10] = 1$ - $P(X < 10) = 1$ - F_X $(10^-) = 0.5$.

- $P[X = 10] = F_X(10+) - F_X(10-) = 0.75$ - $0.5 = 0.25$.
- $P[|X - 5| \leq 0.1] = P[4.9 \leq X \leq 5.1] = P(X \leq 5.1) - P(X < 4.9) = 0.5$ - $0.245 = 0.255$.

  where $F_X(4.9)$ is calculated from the equation of the line $F_X[x] = 0.05x$ when $x < 5$

Problem 3. Exponential random variables have a nice property termed the "memoryless property", which means that $P[T > s + t|T > s] = P[T > t]$, where T is exponentially distributed. Show this property.

Solution 3: The aim is to prove that exponential random variables have property of being memoryless i.e $P[T > s + t|T > s] = P[T > t]$.

Using Bayes Rule

$$P[T > s + t|T > s] = \frac{P[T > s + t, T > s]}{P[T > s]} \tag{14}$$

$$= \frac{P[T > s + t]}{P[T > s]} \tag{15}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \tag{16}$$

$$= e^{-\lambda t} = P[T > t] \tag{17}$$

Problem 4. Suppose $Y = X^2$, where $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 2, \sigma^2 = 3$. Find the pdf of Y. *(HINT: $\frac{d}{dx}\Phi(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$.)*

Solution 4: The RV $Y = g(X)$. The RV X can take on all real values and Y take on only values over $\mathbb{R}+$

- if $y < 0$ then $F_Y(y) = 0$ since $P(Y \geq 0) = 1$
- if $y \geq 0$ then

$$F_Y(y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) \tag{18}$$

$$= P(\frac{-\sqrt{y}-2}{\sqrt{3}} \le \frac{X-2}{\sqrt{3}} \le \frac{\sqrt{y}-2}{\sqrt{3}}) \quad = \Phi(\frac{\sqrt{y}-2}{\sqrt{3}}) - \Phi(\frac{-\sqrt{y}-2}{\sqrt{3}}) \tag{19}$$

Taking the derivative with respect to y, using the chain rule and the hint that $\frac{d}{dx}\Phi(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{24\pi y}}[exp(-\frac{(\sqrt{y}-2)^2}{6}) + exp(-\frac{(\sqrt{y}+2)^2}{6})] & y \ge 0 \\ 0 & y < 0 \end{cases}$$