ECE 341: Probability and Random Processes for Engineers, Spring 2012
Homework 5
Name:
Assigned: 02.08.2012
Due: 02.15.2012

Problem 1. A binary transmission system send a "0" bit by transmitting a $-v$ voltage signal, and a "1" bit by transmitting a $+v$ voltage signal. The received signal is corrupted by Gaussian noise and given by:

$$Y = X + N$$

where X is the transmitted signal (either $+v$ or $-v$), and N is a noise voltage $\sim \mathcal{N}(0, \sigma^2)$. Assume that $P[\text{“}1''] = p = 1 - P[\text{“}0'']$. Find the pdf of Y.

Solution 1: A good approach is to start by evaluating the CDF of Y and express the CDF of Y in terms of CDF of N since it is given to be Random variable with normal distribution. Let A be the event that "0" is transmitted' and B the event that "1' is transmitted.

$$F_Y(y) = F_Y(y|A)P[A] + F_Y(y|B)P[B] \tag{1}$$
$$= P(Y \le y|X = -v)(1-p) + P(Y \le y|X = v)p \tag{2}$$

where $F_Y(y|A)$ is the conditional CDF. Keeping in mind that $Y = X + N$,
$\{Y \le y|X = v\}$ => $\{X + N \le y|X = v\}$ => $\{v + N < y\}$ =>$\{N < y - v\}$
$\{Y \le y|X = -v\}$ => $\{X + N \le y|X = -v\}$ => $\{-v + N < y\}$ =>$\{N < y + v\}$

Thus the CDF of Y can be expressed as the CDF of N. Substitute back in equation 2

$$F_Y(y) = F_N(y + v)(1-p) + F_N(y - v)p \tag{3}$$

The PDF of RV Y is the derivative of the CDF with respect to y

$$f_Y(y) = \frac{d}{dy}F_Y(y) \tag{4}$$
$$= \frac{d}{dy}[F_N(y + v)(1-p) + F_N(y - v)p] \tag{5}$$
$$= f_N(y + v)(1-p) + f_N(y - v)p \tag{6}$$

The Gaussian RV has PDF

$$f_N(n) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp^{\frac{-n^2}{2\sigma^2}} \tag{7}$$

Now we can evaluate the PDF $f_N(y + v)$ and $f_N(y - v)$ by just subtitution of the realization n by $y + v$ and $y - v$

$$f_N(y - v) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp^{\frac{-(y-v)^2}{2\sigma^2}} \tag{8}$$
$$f_N(y + v) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp^{\frac{-(y+v)^2}{2\sigma^2}} \tag{9}$$

Thus

$$f_Y(y) = \frac{1}{\sqrt(2\pi\sigma^2)} \exp^{\frac{-(y+v)^2}{2\sigma^2}} (1-p) + \frac{1}{\sqrt(2\pi\sigma^2)} \exp^{\frac{-(y-v)^2}{2\sigma^2}} p \tag{10}$$

Problem 2. A factory has two spares of a critical system component that has an average lifetime of $1/\lambda = 1$ month. Find the probability that the three components (the operating one and the two spares) will last more than 6 months. Assume the component lifetimes are exponential random variables.

Solution 2: The remaining lifetime of the component in service is an exponential random variable with rate λ by the memoryless property. Thus, the total lifetime X of the three components is the sum of the three random variables with parameter $\lambda = 1$. Thus X has a 3-Erlang distribution with $\lambda = 1$. The probability that X is greater than 6 is

$$P(X > 6) = 1 - P(x \geq 6) \tag{11}$$

$$= \sum_{k=0}^{k=2} \frac{6^k}{k!} \exp^{-6} = 0.06197 \tag{12}$$

Problem 3. Let $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$. Find the pdf of $Y = aX + b$ for a, b known constants.

Solution 3:

$$F_Y(y) = P(Y \leq y) \tag{13}$$

$$= P(aX + b \leq y) = P(X \leq \frac{y-b}{a}) = F_X(\frac{y-b}{a}) \tag{14}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\frac{y-b}{a}) = \frac{1}{a} f_X(\frac{y-b}{a}) \tag{15}$$

$$\tag{16}$$

The PDF of X is

$$f_X(x) = \frac{1}{\sqrt(2\pi\sigma^2)} \exp^{\frac{-(x-\mu)^2}{2\sigma^2}} \tag{17}$$

Substituting the realization x by $\frac{y-b}{a}$ gives the pdf

$$f_Y(y) = \frac{1}{a} \frac{1}{\sqrt(2\pi\sigma^2)} \exp^{\frac{-(y-b-\mu)^2}{2a^2\sigma^2}} \tag{18}$$

$$= \frac{1}{\sqrt{2\pi a^2 \sigma^2}} \exp^{\frac{-(y-b-\mu)^2}{2a^2\sigma^2}} \tag{19}$$

We thus recognize that $Y = aX + b \sim \mathcal{N}(\mu_x + b, a^2\sigma_x^2)$.

Problem 4. Let X be the uniform random variable in the interval $[-2, 2]$. Find and plot $P[|X| > x]$.
Solution 4:

$$\begin{aligned} P[|X| > x] &= P[\{X > x\} \cup \{X < -x\}] && (20) \\ &= P[X > x] + P[X < -x] && (21) \\ &= 1 - F_x(x) + F_X(-x) && (22) \end{aligned}$$

But

$$f_X(x) = \begin{pmatrix} \frac{1}{4} & -2 \le x \le 2 \\ 0 & otherwise \end{pmatrix}$$

$$F_X(x) = \begin{pmatrix} 0 & x \le -2 \\ \frac{1}{4}(x+2) & -2 \le x \le 2 \\ 1 & x \ge 2 \end{pmatrix}$$

$$P[|X| > x] = \begin{pmatrix} 1 & x \le 0 \\ 1 - \left(\frac{x+2}{4}\right) + \left(\frac{-x+2}{4}\right) = 1 - \frac{x}{2} & 0 \le x \le 2 \\ 0 & x \ge 2 \end{pmatrix}$$