ECE 341: Probability and Random Processes for Engineers, Spring 2012
Homework 4
Name:
Assigned: 02.01.2012
Due: 02.08.2012

Problem 1. A Zipf $(n, \alpha = 1)$ random variable X has p.m.f.

$$P_X(x) = c(n)/x, \text{ for } x = 1, 2, 3, \cdots, n$$

The constant $c(n)$ is set so that $\sum_{n=1}^{n} P_X(x) = 1$ (to make it a proper p.m.f.). Calculate $c(n)$ for $n = 1, 2, 3, 4, 5, 6$.

Solution 1: The requirement that

$$\sum_{n=1}^{n} P_X(x) = 1 \tag{1}$$

$$\text{For } n = 1 : c(1) * [\frac{1}{1}] = 1 ==> c(1) = 1; \tag{2}$$

$$\text{For } n = 2 : c(2) * [\frac{1}{1} + \frac{1}{2}] = 1 ==> c(2) = \frac{2}{3}; \tag{3}$$

$$\text{For } n = 3 : c(3) * [\frac{1}{1} + \frac{1}{2} + \frac{1}{3}] = 1 ==> c(3) = \frac{6}{11}; \tag{4}$$

$$\text{For } n = 4 : c(4) * [\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}] = 1 ==> c(4) = \frac{12}{25}; \tag{5}$$

$$\text{For } n = 5 : c(5) * [\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}] = 1 ==> c(5) = \frac{60}{137}; \tag{6}$$

$$\text{For } n = 6 : c(6) * [\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}] = 1 ==> c(6) = \frac{20}{49}; \tag{7}$$

Problem 2. The Zipf $(n, \alpha = 1)$ random variable X introduced in Problem 1 is often used to model the "popularity" of a collection of n objects. For example, a Web server can deliver one of n Web pages. The pages are numbered such that the page 1 is the most requested page, page 2 is the second most requested page and so on. If page k is requested, then $X = k$. To reduce external network traffic, an ISP gateway caches copies of the k most popular pages. Using Matlab, calculate, as a function of n for $1 \le n \le 100$, how large k must be to ensure that the cache can deliver a page with probability 0.75 or more.

```
function k=zipfcache(n,p);
%Usage: k=zipfcache(n,p);
%for the Zipf (n,alpha=1) distribution, returns the smallest k
%such that the first k items have total probability p
pmf=1./(1:n);
pmf=pmf/sum(pmf); %normalize to sum to 1
cdf=cumsum(pmf);
k=1+sum(cdf<=p);
```

Solution 2: Suppose X_n is a Zipf $(n, \alpha = 1)$ random variable and thus has PMF

$$P_X(x) = \begin{pmatrix} \frac{c(n)}{x} & x = 1, 2, ...n \\ 0 & otherwise \end{pmatrix}$$

The problem asks us to find the smallest value of k such that $P[X_n \leq k] \geq 0.75$. That is, if the server caches the k most popular files, then with $P[X_n \leq k]$ the request is for one of the k cached files. First, we might as well solve this problem for any probability p rather than $p = 0.75$. Thus, in math terms, we are looking for

$$k = \min\{k' | P[X_n \leq k'] \geq p\} \tag{8}$$

What makes the Zipf distribution hard to analyze is that there is no closed form expression for

$$c(n) = (\sum_{x}^{n} (\frac{1}{x}))^{-1} \tag{9}$$

Thus we use MATLAB to grind through calculations. The following simple program generates the Zipf distribution and returns the correct value of k.

The program zipfcache generalizes 0.75 to be the probability p. Although this program is sufficient the problem asks us to find k for all values of n from 1 to 10^2. One way to do this is to call Zipfcache a hundred times to find k for each value of n. A better way is to use the properties of the Zipf PDF. In particular

$$P[X_n \leq k'] = c(n) \sum_{K=1}^{k'} (\frac{1}{x}) = \frac{c(n)}{c(k')} \tag{10}$$

```
function k=zipfcacheall(n,p);
%Usage: k=zipfcacheall(n,p);
%returns vector k such that the first
%k(m) items have total probability >= p
%for the Zipf(m,1) distribution.
c=1./cumsum(1./(1:n));
k=1+countless(1./c,p./c);
```

Thus we wish to find

$$k = \min\{k' | \frac{c(n)}{c(k')}) \geq p\} = \min\{k' | \frac{1}{c(k')} \geq \frac{p}{c(n)}\} \tag{11}$$

Note that the definition of k implies that

$$\frac{1}{c(k')} < \frac{p}{c(n)} \tag{12}$$

$$\tag{13}$$

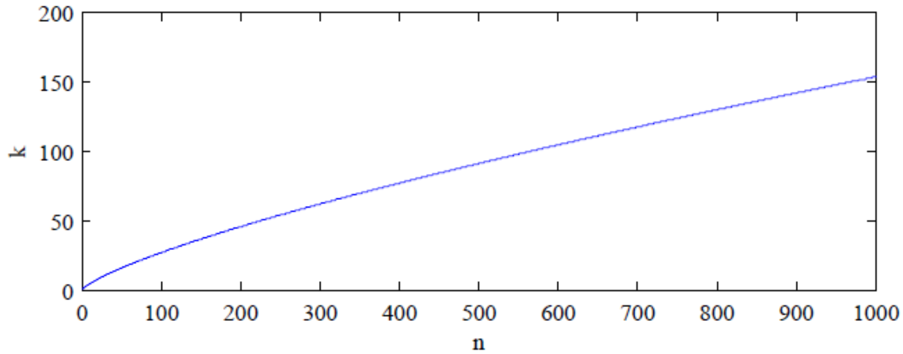
$$\text{for } k' = 1, ..., k-1. \tag{14}$$

Using the notation $|A|$ to denote the number of elements in the set A, we can write

$$k = 1 + |k' \{\frac{1}{c(k')} < \frac{p}{c(n)}\}| \tag{15}$$

This is the basis for a very short MATLAB program:

Note that zipfcacheall uses a short Matlab program countless.m that is almost the same as count.m introduced in Example 2.47. If n=countless(x,y), then $n(i)$ is the number of elements of x that are strictly less than $y(i)$ while count returns the number of elements less than or equal to $y(i)$. In any case, the commands k=zipfcacheall(100,0.75); plot(1:100,k); is sufficient to produce this figure of k as a function of m:



We see in the figure that the number of files that must be cached grows slowly with the total number of files n.

Finally, we make one last observation. It is generally desirable for Matlab to execute operations in parallel. The program zipfcacheall generally will run faster than n calls to zipfcache. However, to do its counting all at once, countless generates and nxn array. When n is not too large, say $n \leq 100$, the resulting array with $n^2 = 1{,}0000$ elements fits in memory. For much large values of n, say $n = 106$ (as was proposed in the original printing of this edition of the text), countless will cause an "out of memory" error.

Problem 3. We measure for resistance R of each resistor in a production line and we accept only the units whose resistance is between 96 and 104 ohms. Find the percentage of the accepted units if

- R is uniform between 95 and 105 ohms
- R is Gaussian with mean 100 and standard deviation 2 ohms.

Solution 3: Percentage of units between 96 and 104 ohms equals 100 p and p is calculated as follows

$$P = P(95 \leq X \leq 104) = F(104) - F(96) \quad (16)$$

where F(.) is the cumilitive density function

Case 1: R is uniform between 95 and 105 ohms

$$F(X) = 0.1(X - 95) \text{ for } 95 \leq X \leq 100 \tag{17}$$

$$P = 0.1(104 - 95) - 0.1(96 - 95) = 0.8 \tag{18}$$

Case 2: R is Gaussian with mean 100 and standard deviation 2 ohms.

$$P = \Phi\left(\frac{104 - 100}{2}\right) - \Phi\left(\frac{96 - 100}{2}\right) = 0.987 \tag{19}$$

Problem 4. The probability of heads of a random coin is a RV P uniform in the interval $(0, 1)$.

- Find the probability $P[0.3 \leq P \leq 0.7]$.
- The coin is tossed 10 times and heads shows 6 times. Given this fact, find the probability that P is between 0.3 and 0.7.

Solution 4:

Find the probability $P[0.3 \leq P \leq 0.7]$

$$P[0.3 \leq P \leq 0.7] = \int_{0.3}^{0.7} dp = 0.4 \tag{20}$$

The coin is tossed 10 times and heads shows 6 times. Given this fact, find the probability that P is between 0.3 and 0.7. We have that head shows 6 times from 10 tosses and we are asked to get the conditional probability $P[0.3 \leq P \leq 0.7|A]$ where A is the event you get 6 times heads out of ten tosses

$$f(p|A) = \frac{(p^6)(1-p)^4}{\int_0^1 p^6(1-p)^4\, dp} = \frac{(p^6)(1-p)^4}{4329 * 10^{-7}} \tag{21}$$

$$\tag{22}$$

$$P[0.3 \leq P \leq 0.7|A] = \int_{0.3}^{0.7} f(p|A)\, dp = \frac{10^7}{4329} \int_{0.7}^{0.3} p^6(1-p)^4\, dp = 0.768 \tag{23}$$