

## ECE 341: Probability and Random Processes for Engineers, Spring 2012

Homework 3

**Name:**

Assigned: 01.25.2012

Due: 02.01.2012

**Problem 1.** You are the manager of a ticket agency that sells concert tickets. You assume that people will call three times in an attempt to buy tickets, and then give up. You want to make sure that you are able to serve at least 95% of the people who want tickets. Let  $p$  be the probability that a caller gets through to your ticket agency. What is the minimum value of  $p$  necessary to meet your goal?

*Solution 1:* The probability of success for a caller to call one time and gets through is  $p$ . The probability of fail i.e the caller doesn't get through to your ticket agency is  $1 - p$ . For that failure to happen three times then this would happen with probability of  $(1 - p)^3$ . We want this probability of failure to be  $\leq 0.05$ .

$$(1 - p)^3 \leq 0.05 \Rightarrow p = 0.6316. \quad (1)$$

**Problem 2.** The UIC Flames and the Northwestern Wildcats are in a best out of five game playoff series – they play games until one of the teams has won three games and is declared the winner. Assume that either team is equally likely to win any game independently of any other game played (an unrealistic assumption...). Find:

- The p.m.f.  $P_N(n)$  for the total number  $N$  of games played in the series.
- The p.m.f.  $P_F(f)$  for the number  $F$  of Flames wins in the series.
- The p.m.f.  $P_W(w)$  for the number  $W$  of Wildcats losses in the series.

*Solution 2:*

- The p.m.f.  $P_N(n)$  for the total number  $N$  of games played in the series

Since the two teams are equally likely to win then the probability of winning a game for any team is 0.5. Let  $A_n$  denote the event that Northwestern Wildcats wins the playoff series. Let  $B_n$  be the event that UIC Flames win the playoff series. Note that  $n$  is the number of games needed to have a winner team. To win the series any team should win 3 out of 5 games. This could happen in 3 ways. Choose any team and get the probability of win because both have the same probability.

Case1: Wildcats win in 3 straight games.

$$P(A_3) = \frac{1^3}{2^3} = \frac{1}{8} \quad (2)$$

Case2: Wildcats win the series in total of 4 games keeping in mind that the 3 games won should not be straight or else it is case 1 this happens if they win 2 out of first 3 and they win the fourth

$$P(A_4) = \binom{3}{2} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right) = \frac{3}{16} \quad (3)$$

Case3: Wildcats win 3 out of total of 5 games this happens if they win two out of the first four games and then win game number five

$$P(A_5) = \binom{4}{2} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right) = \frac{3}{16} \quad (4)$$

The question asks for  $P_N(n)$  (p.m.f) for  $N$  played which happens if either UIC Flames or Northwestern wildcats win. In this case we have two teams with equal probability of win thus  $P_N(n) = P(A_n) + P(B_n) = 2P(A_n)$  Thus the total number of games,  $N$ , played in 5 series is described by PMF

$$P_N(n) = \begin{pmatrix} 2\frac{1^3}{2} = \frac{1}{4} & n = 3 \\ 2\binom{3}{2}\frac{1^3}{2}\frac{1}{2} = \frac{3}{8} & n = 4 \\ 2\binom{4}{2}\frac{1^4}{2}\frac{1}{2} = \frac{3}{8} & n = 5 \\ 0 & \text{otherwise} \end{pmatrix}$$

- For the total number of UIC flames wins  $F$ , we note that if UIC flames get  $f < 3$ , then the Northwestern won the  $3+f$  games. Also the UIC flames win 3 games if they win the series in 3,4,5 games.

$$P[F = f] = \begin{pmatrix} P(A_{3+f}) & f = 0, 1, 2 \\ P(B_3) + P(B_4) + P(B_5) & f = 3 \end{pmatrix}$$

Thus the number of wins by the UIC flames,  $F$ , has PMF shown below.

$$P[F = f] = \begin{pmatrix} P(A_3) = \frac{1}{8} & f = 0 \\ P(A_4) = \frac{3}{16} & f = 1 \\ P(A_5) = \frac{3}{16} & f = 2 \\ \frac{1}{8} + \frac{3}{16} + \frac{3}{16} = \frac{1}{2} & f = 3 \\ 0 & \text{otherwise} \end{pmatrix}$$

- The number of Wildcats loss  $W$  is equal to the number of UIC Flames wins  $F$ . This implied  $P_W(w) = P_F(w)$ . Since either team is equally likely to win any game, by symmetry,  $P_F(w) = P_W(w)$ . This implies  $P_W(w) = P_F(w)$ . The complete expression of the PMF of

$$P_W(w) = P_F(w) = \begin{pmatrix} \frac{1}{8} & f = 0 \\ \frac{3}{16} & f = 1 \\ \frac{3}{16} & f = 2 \\ \frac{1}{2} & f = 3 \\ 0 & \text{otherwise} \end{pmatrix}$$

**Problem 3.** Suppose a cellular phone costs \$20 per month with 30 minutes of use included and that each additional minute of use costs \$0.50. If the number of minutes you use in a month is a geometric random variable  $M$  with expected value of  $E[M] = 1/p = 30$  minutes, what is the p.m.f. of  $C$ , the cost of the phone for one month?

*Solution 3:* The cellular calling plan charges a flat rate of 20 dollar per month up to and including the 30th minute, and an additional 50 cents for each minute over 30 minutes. Knowing that the time you spend of the phone is geometric random variable  $M$  with mean  $\frac{1}{p} = 30$ , the PMF of  $M$  is

$$P_M(m) = \begin{pmatrix} (1-p)^{m-1}p & m = 1, 2, \dots \\ 0 & \text{otherwise} \end{pmatrix}$$

When  $M \leq 30$ , the monthly cost,  $C$  obeys

$$P_C(20) = P[M \leq 30] = \sum_{m=1}^{30} (1-p)^{m-1}p = 1 - (1-p)^{30} \quad (5)$$

When  $M > 30$ ,  $C = 20 + \frac{(M-30)}{2}$  or  $M = 2C - 10$ . Thus

$$P_C(c) = P_M(2c - 10) \quad c = 20.5, 21, 21.5, \dots \quad (6)$$

The complete PMF of  $C$  is

$$P_C(c) = \begin{pmatrix} 1 - (1-p)^{30} & c = 20 \\ (1-p)^{2c-10-1}p & c = 20.5, 21, 21.5, \dots \end{pmatrix}$$

**Problem 4.** The binomial random variable  $X$  has p.m.f.

$$P_X(x) = \binom{5}{x} (1/2)^5$$

- Find the standard deviation of the random variable  $X$ .
- What is  $P[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X]$ , the probability that  $X$  is within one standard deviation of the expected value?

*Solution 4:*

- The Expected Value of  $X$  is

$$\begin{aligned} E[X] &= \sum_{x=0}^5 xP_X(x) \\ &= 0 \binom{5}{0} \frac{1}{2^5} + 1 \binom{5}{1} \frac{1}{2^5} + 2 \binom{5}{2} \frac{1}{2^5} + 3 \binom{5}{3} \frac{1}{2^5} + 4 \binom{5}{4} \frac{1}{2^5} + 5 \binom{5}{5} \frac{1}{2^5} \\ &= \frac{[5 + 20 + 30 + 20 + 5]}{2^5} = \frac{5}{2} \end{aligned}$$

- The Expected Value of  $X^2$  is

$$\begin{aligned} E[X^2] &= \sum_{x=0}^5 x^2 P_X(x) \\ &= 0^2 \binom{5}{0} \frac{1}{2^5} + 1^2 \binom{5}{1} \frac{1}{2^5} + 2^2 \binom{5}{2} \frac{1}{2^5} + 3^2 \binom{5}{3} \frac{1}{2^5} + 4^2 \binom{5}{4} \frac{1}{2^5} + 5^2 \binom{5}{5} \frac{1}{2^5} \\ &= \frac{[5 + 40 + 90 + 80 + 25]}{2^5} = \frac{15}{2} \end{aligned}$$

- The variance of  $X$  is

$$\text{var}[X] = E[X^2] - (E[X])^2 = \frac{15}{2} - \frac{25}{4} = \frac{5}{4} \quad (7)$$

By taking the square root of the variance, the standard deviation of  $X$  is  $\sigma_X = \sqrt{\frac{5}{4}} = 1.12$

The probability that  $X$  is within one standard deviation of its mean is

$$\begin{aligned} P[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X] &= P[2.5 - 1.12 \leq X \leq 2.5 + 1.12] \\ &= P[1.38 \leq X \leq 3.62] \\ &= P[2 \leq X \leq 3] \end{aligned}$$

By summing the PMF over the desired range, we obtain

$$P[2 \leq X \leq 3] = P_X(2) + P_X(3) = \frac{10}{32} + \frac{10}{32} = \frac{5}{8} \quad (8)$$