

$$P(X = 0|Y = 1) = \frac{P(X = 0, Y = 1)}{P(Y = 1)} = \frac{P(Y = 1|X = 0)P(X = 0)}{P(Y = 1)} = \frac{p\epsilon_1}{p\epsilon_1 + 1 - \epsilon_2 - p + p\epsilon_2} \quad (5)$$

$$P(X = 1|Y = 1) = \frac{P(X = 1, Y = 1)}{P(Y = 1)} = \frac{P(Y = 1|X = 1)P(X = 1)}{P(Y = 1)} = \frac{(1-p)(1-\epsilon_2)}{p\epsilon_1 + 1 - \epsilon_2 - p + p\epsilon_2} \quad (6)$$

(7)

We just need to compare the Numerator of the two expressions i.e $p\epsilon_1$ and $(1-p)(1-\epsilon_2)$ (8)

We dont need to compare the denominator because the denominator is the same (9)

However $P(Y = 1)$ is shown below (10)

$$P(Y = 1) = P(X = 0, Y = 1) + P(X = 1, Y = 1) \quad (11)$$

$$P(Y = 1) = P(X = 0)P(Y = 1|X = 0) + P(X = 1)P(Y = 1|X = 1) \quad (12)$$

$$P(Y = 1) = p\epsilon_1 + (1-p)(1-\epsilon_2) \quad (13)$$

$$P(Y = 1) = p\epsilon_1 + 1 - \epsilon_2 - p + p\epsilon_2 \quad (14)$$

(15)

- Set $\epsilon_1 = \epsilon_2 = \epsilon$. For what value of ϵ are the inputs and outputs independent?

For inputs and outputs to be independent then $P(Y = 0|X = 0) = P(Y = 0|X = 1) = P(Y = 0)$

$$P(Y = 0|X = 0) = P(Y = 0|X = 1) = P(Y = 0) \Rightarrow 1 - \epsilon = \epsilon \Rightarrow \epsilon = 0.5 \quad (16)$$

Problem 2. Players X and Y roll dice alternatively starting with X. The player that rolls eleven wins. Show that the probability that player X wins is $18/35$. *HINT: Let $A = \{\text{event that X wins}\}$, $M = \{\text{eleven shows at first try}\}$, and use $P[A] = P[A|M]P[M] + P[A|M^c]P[M^c]$.*

Solution 2:

Let us compute each term from the $P[A] = P[A|M]P[M] + P[A|M^c]P[M^c]$

$$\star P(M) = \frac{2}{36}; \text{ Reason :The two possible dice values that gives sum of 11 is (6,5) or (5,6)} \quad (17)$$

(18)

$$\star P(M^c) = 1 - P(M) = \frac{34}{36} \quad (19)$$

(20)

$$\star P(A|M) = 1; \text{ Reason: takes place only if X wins since he starts rolling the dice.} \quad (21)$$

(22)

$$\star P(A|M^c) = \text{Probability that X wins but not from the first time which is equivalent} \quad (23)$$

to probability of player Y winning instead. However the Probability of player Y winning is (24)

$$P(A^c) = 1 - P(A) \Rightarrow P(A|M^c) = P(A^c) = 1 - P(A) \quad (25)$$

(26)

$$\star P(A) = P(A|M)P(M) + P(A|M^c)P(M^c) = \frac{2}{36} + (1 - P(A)) * \frac{34}{36} \Rightarrow P(A) = \frac{18}{35} \quad (27)$$

Problem 3. A student needs eight chips of a certain type to build a circuit. It is known that 5% of these chips are defective. How many chips should he buy for there to be a greater than 90% probability of having enough chips for the circuit?

Solution 3:

The probability that k chips out of n chips are not defective is

$$P = \binom{n}{k} * p^k * (1 - p)^{n-k} \quad p \text{ probability that chip is not defective} \quad (28)$$

Let A be the event that 8 out of these 8 chips are not defective

$$P(A) = \binom{8}{8} * 0.95^8 * 0.05^0 = 0.663424 \quad (29)$$

Let us try to buy 9 chips instead of 8 and calculate the probability of having those 9 not defective or **8 out of these 9**. Let B be the event that 9 out of these 9 chips are not defective Let C be the event that 8 out of these 9 chips are not defective

$$P(B) = \binom{9}{9} * 0.95^9 = 0.63024 \tag{30}$$

$$P(C) = \binom{9}{8} * 0.95^8 * 0.05^1 = 0.298 \tag{31}$$

let S be the event of having 8 or 9 chips not defective (32)

$$P(S) = P(B) + P(C) = 0.928 \tag{33}$$

Thus 9 chips are good enough (34)

Problem 4. Consider a well shuffled deck of 52 cards of which four are aces and four are kings.

- What is the probability of obtaining an ace in the 1st draw?
- Draw a card and look at it. Do not replace. What is the probability of drawing an ace in the 2nd draw? Does the answer change if you had not looked at the 1st draw?
- Suppose we draw 7 cards. What is the probability that the seven cards include three aces? That they include two kings?
- Suppose the entire deck of cards is distributed equally among four players. What is the probability that each player gets an ace?

Solution 4:

- Name Event of obtaining Ace as A from first time

$$P(A) = \frac{4}{52} \tag{35}$$

$$\tag{36}$$

- Event of having picked Ace first time **A1** and Event of having picked Ace for second time **A2**

First Case: Card drawn, looked at and not replaced
 If Card 1 was an Ace then

$$p(A2|A1) = \frac{3}{51} \tag{37}$$

If Card 1 was an Not Ace then

$$p(A2|A1^c) = \frac{4}{51} \tag{38}$$

Second Case: Card drawn and not looked at

$$P(A2) = P(A2|A1) * P(A1) + P(A2|A1^c) * P(A1^c) = \frac{3}{51} * \frac{4}{52} + \frac{4}{51} * \frac{48}{52} = \frac{1}{13} \tag{39}$$

$$\tag{40}$$

- The concept of $\binom{n}{k}$ should come up in mind when solving such a problem

Name the event of having 3 Aces from 7 drawn cards as **A**.

The fact that we need to need 3 Aces out of 4 available Aces is translated as $\binom{4}{3}$

The fact that we need to need the rest of the cards to be anything else other than Aces and Kings is translated as $\binom{48}{4}$

The total number of cases is translated as $\binom{52}{7}$

$$P(A) = \frac{\binom{4}{3} \binom{48}{4}}{\binom{52}{7}} = 0.00582 \tag{41}$$

Name the event of having 2 Kings from 7 drawn cards as **B**.

The fact that we need to need 2 kings out of 4 available kings is translated as $\binom{4}{2}$

The fact that we need to need the rest of the cards to be anything else other than Aces and Kings is translated as $\binom{48}{5}$

The total number of cases is translated as $\binom{52}{7}$

$$P(B) = \frac{\binom{4}{2} \binom{48}{5}}{\binom{52}{7}} = 0.07679 \tag{42}$$

$$\tag{43}$$

As for the probability of A or B taking place can be calculated as

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

And $P(A \cap B)$ is calculated as follows

The fact that we need to need 2 kings out of 4 available kings is translated as $\binom{4}{2}$

The fact that we need to need 3 Aces out of 4 available aces is translated as $\binom{4}{3}$

The fact that we need the rest 2 cards to be anything else other than Aces and Kings is translated as $\binom{44}{2}$

The total number of cases is translated as $\binom{52}{7}$ Thus

$$P(A \cap B) = \frac{\binom{4}{3} \binom{4}{2} \binom{44}{2}}{\binom{52}{7}} = 0.00017 \quad (44)$$

$$P(A \cup B) = 0.00582 + 0.07679 - 0.00017 = 0.0824 \quad (45)$$

• The cards are to be distributed to 4 players then each player would get 13 cards thus there are total of $52!$ permutations of all the cards. Since each player receives 13 cards, and order does not matter within a hand, there are $\frac{52!}{(13!)^4}$ ways of distributing the cards on 4 players, or this many “hands”. There are $4!$ ways of arranging the 4 Aces and giving each player one of them As for the remaining 48 cards, there are $\frac{48!}{(12!)^4}$ ways of distributing the cards between the 4 players Let the A be the event of having one ace for each player then

$$P(A) = \frac{\frac{48!}{(12!)^4} * 4!}{\frac{52!}{(13!)^4}} = 0.1055 \quad (46)$$

$$(47)$$