

ECE 341: Probability and Random Processes for Engineers, Spring 2012

Homework 13 - Last homework

Name:

Assigned: 04.18.2012

Due: 04.25.2012

Problem 1. Let $X(t)$ be the input to a linear time-invariant filter. We are given that $X(t)$ is a wide sense stationary stochastic process with expected value $\mu_X = 4$ volts. The filter output $Y(t)$ is also a wide sense stationary stochastic process with expected value $\mu_Y = 1$ volt. The filter impulse response is $h(t) = e^{-t/a}$ for $t \geq 0$ and 0 otherwise. What is the value of the time constant a ?

Solution 1: The mean of the output is

$$\mu_Y = \mu_X \int_{-\infty}^{+\infty} h(t) dt \quad (1)$$

$$= 4 \int_0^{+\infty} e^{-t/a} dt \quad (2)$$

$$= -4ae^{-t/a} \Big|_0^{+\infty} \quad (3)$$

$$= 4a \rightarrow a = \frac{1}{4}. \quad (4)$$

Problem 2. A wide sense stationary process $X(t)$ with autocorrelation function $R_X(\tau)$ and power spectral density $S_X(f)$ is the input to a tapped delay line filter (this is a linear time invariant filter) with frequency response

$$H(f) = a_1 e^{-j2\pi f t_1} + a_2 e^{-j2\pi f t_2}$$

Solution 2: Find the output power spectral density $S_Y(f)$ and the output autocorrelation $R_Y(\tau)$. Since $S_Y(f) = S_X(f)|H(f)|^2$

$$|H(f)|^2 = H(f)H(f^*) \quad (5)$$

$$= (a_1 e^{-j2\pi f t_1} + a_2 e^{-j2\pi f t_2})(a_1 e^{+j2\pi f t_1} + a_2 e^{+j2\pi f t_2}) \quad (6)$$

$$= (a_1^2 + a_2^2)S_X(f) + a_1 a_2 S_X(f) e^{-j2\pi f(t_1 - t_2)} + a_1 a_2 S_X(f) e^{-j2\pi f(t_2 - t_1)} \quad (7)$$

By inverse fourier then

$$R_Y(\tau) = (a_1^2 + a_2^2)R_X(\tau) + a_1 a_2 (R_X(\tau - (t_1 - t_2)) + R_X(\tau + (t_1 - t_2))) \quad (8)$$

Problem 3. A wide sense stationary process $X(t)$ with autocorrelation function $R_X(\tau) = e^{-4\pi\tau^2}$ is the input to a filter with transfer function $H(f) = 1$ for $0 \leq |f| \leq 2$ (and 0 else). Find:

1. The average power of the input $X(t)$
2. The output power spectral density $S_Y(f)$.
3. The average power of the output $Y(t)$.

Solution 3:

1. The average power of the input is

$$E[X^2(t)] = R_X(0) = 1 \tag{9}$$

2. The input has power spectral density

$$S_X(f) = \frac{1}{2} e^{-\frac{\pi f^2}{4}} \tag{10}$$

The output PSD $S_Y(f) = S_X(f)|H(f)|^2$ then,

$$S_Y(f) = \begin{pmatrix} \frac{1}{2} e^{-\frac{\pi f^2}{4}} & |f| \leq 2 \\ 0 & \text{otherwise} \end{pmatrix}$$

3. The average output power is

$$E[Y^2(t)] = \int_{-\infty}^{+\infty} S_Y(f) df = \frac{1}{2} \int_{-2}^2 e^{-\frac{\pi f^2}{4}} df \tag{11}$$

This integral cannot be expressed in closed form. However, we can express it in the form of the integral of a standardized Gaussian PDF by making the substitution $f = z\sqrt{2\pi}$. With this substitution,

$$E[Y^2(t)] = \frac{1}{2\pi} \int_{-\sqrt{2\pi}}^{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \tag{12}$$

$$= \Phi(\sqrt{2\pi}) - \Phi(-\sqrt{2\pi}) \tag{13}$$

$$= 2\Phi(\sqrt{2\pi}) - 1 = 0.9876 \tag{14}$$

The output power almost equals the input power because the filter bandwidth is sufficiently wide to pass through nearly all of the power of the input.

Problem 4. Let X be a random variable (an i.i.d. source) with 7 outputs with probability mass function given by $[0.49, 0.26, 0.12, 0.04, 0.04, 0.03, 0.02]$ (the i -th element of this vector is the probability of the i -th symbol).

1. Find the entropy of this source.
2. Construct a Huffman code for this source.

3. What is the efficiency of the code you constructed in part 2?

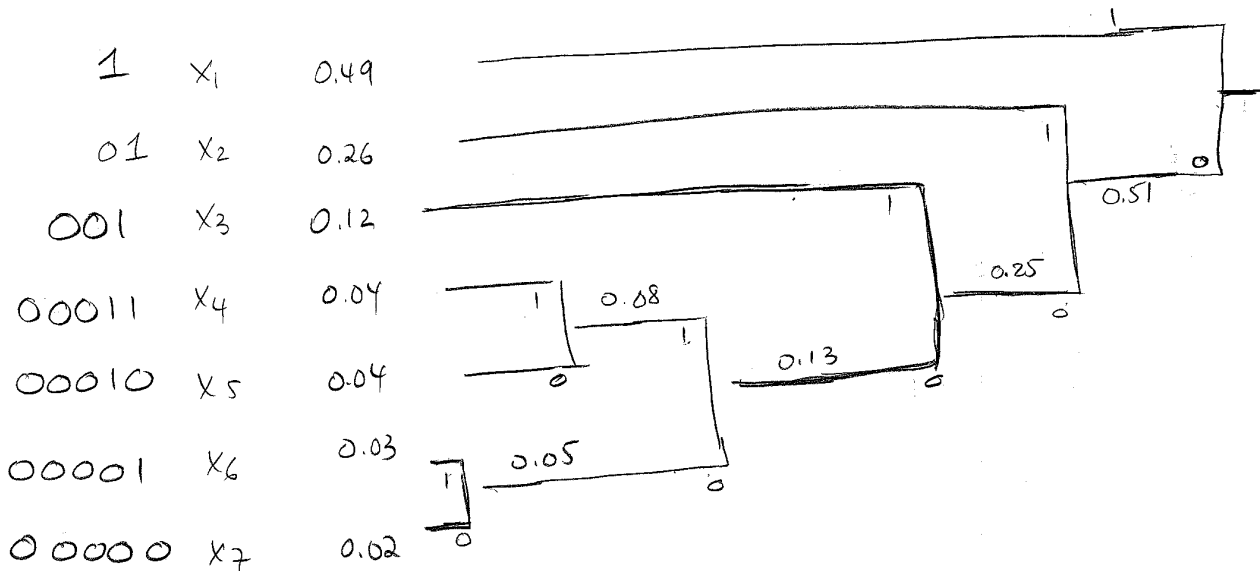
Solution 4:

1. The entropy of this source $H(X)$

$$H(X) = - \sum_i p(x_i) \log p(x_i) = -(0.49 \log 0.49 + 0.26 \log 0.26 + 0.12 \log 0.12 + 0.04 \log 0.04) \quad (15)$$

$$+ 0.04 \log 0.04 + 0.03 \log 0.03 + 0.02 \log 0.02 = 2.01 \text{bits} \quad (16)$$

$$(17)$$



Problem 4

The expected length is

$$L(C) = \sum_i l_i p(x_i) = 0.49 + 2(0.26) + 3(0.12) + 5(0.04) + 5(0.04) + 5(0.03) + 5(0.02) = 2.02 \text{bits} \quad (18)$$

where l_i is the length of the codeword assigned to the symbol.

3. The efficiency is

$$\eta = \frac{H(X)}{L(C)} = \frac{2.01}{2.02} = 0.9950 \quad (19)$$

$$= 99.5\% \quad (20)$$

Problem 5. One is given 6 bottles of wine. It is known that precisely one bottle has gone bad (it tastes bad). From inspection of the bottles it is determined that the probability p_i that the i -th bottle is bad is given by $(p_1, p_2, \dots, p_6) = (\frac{8}{23}, \frac{6}{23}, \frac{4}{23}, \frac{2}{23}, \frac{2}{23}, \frac{1}{23})$. Tasting will determine the bad wine.

Suppose you taste the wines one at a time. Choose the order of tasting to minimize the expected number of tasting required to determine the bad bottle. Remember, if the first 5 wines pass the test you don't have to taste the last.

1. What is the expected number of tastings required?
2. Which bottle should be tasted first?

Now you get smart. For the first sample, you mix some of the wines in a fresh glass and sample the mixture. You proceed, maxing and tasting, stopping when the bad bottle has been determined.

3. What is the minimum expected number of tastings required to determine the bad wine?
4. What mixture should be tasted first?

Solution 5:

1. Given the probability of the i th bottle that might have the bad wine, we should start tasting the bottles in decreasing probability order, starting from the one with higher probability and so on.

$$N = 1 \frac{8}{23} + 2 \frac{6}{23} + 3 \frac{4}{23} + 4 \frac{2}{23} + 5 \frac{2}{23} + 5 \frac{1}{23} = \frac{55}{23} \tag{21}$$

Notice that the wine with the least probability is to be multiplied by 5 since if the first 5 wines pass the test you don't have to taste the last.

2. The bottle that should be tasted first is the one with the higher probability of containing the bad wine $p_1 = \frac{8}{23}$.

3. Since we know that the Huffman code provides the minimum expected length, we can use it for solving this problem. Then, we compute the expected length as follows:

$$L(C) = \sum_i l_i p(x_i) = 2 \frac{8}{23} + 2 \frac{6}{23} + 2 \frac{4}{23} + 3 \frac{2}{23} + 4 \frac{2}{23} + 4 \frac{1}{23} = \frac{54}{23}$$

4. From the diagram, the paths of the Huffman code can be used to make the mixtures by grouping the probabilities, the bottles with probabilities $\frac{8}{23}$ and $\frac{6}{23}$ (the bottles that lead to the sum at the end of $\frac{14}{23}$ probability. This mixture should be tasted first since it has the higher probability.

If the bad wine is not in this mixture then the bad wine must be in one of the remaining bottles (the path with the one leading to the probability $\frac{9}{23}$).

