ECE 341: Probability and Random Processes for Engineers, Spring 2012
Homework 11
Name:
Assigned: 04.04.2012
Due: 04.11.2012

Problem 1. Textbook problem 6.1.3. Do it on your own rather than looking at the solution.

Solution 1:

a-The fact that the first correct answer needs to be on the nth phone call means, there were $n-1$ wrong calls before that

$$P_{N_1}[n] = \begin{pmatrix} (\frac{3}{4})^{n-1}\frac{1}{4} & n = 1, 2, ... \\ 0 & \text{otherwise} \end{pmatrix}$$

b- N_1 is geometric with parameter $p = \frac{1}{4}$. For a geometric RV then mean is $E(N_1) = \frac{1}{p} = 4$.

c- The fact that the fourth is call is the fourth correct answer needs to be on the nth phone call means that there were $n-1$ calls contains three correct answers

$$P_{N_1}[n] = \begin{pmatrix} \binom{n-1}{3}(\frac{3}{4})^{n-4}(\frac{1}{4})^4 & n = 1, 2, ... \\ 0 & \text{otherwise} \end{pmatrix}$$

d- By summing up the means of 4 identically distributed geometric random variables each with mean 4, we get $E[N_4] = 4E[N_1] = 16$.

Problem 2. Let U be a continuous random variable with uniform distribution over $[u_1, u_2]$.

- Find the moment generating function of U.
- Use the MGF to calculate the first moment of U.
- Use the MGF to calculate the second moment of U.

Solution 2: From the definition of the characteristic function

$$\phi_U(s) = E(e^{su}) = \int_{u_1}^{u_2} e^{su}\frac{1}{u_2 - u_1}du = \frac{e^{su_2} - e^{su_1}}{s(u_2 - u_1)} \tag{1}$$

The first moment is

$$E[U] = \left.\frac{d\phi_U(s)}{ds}\right|_{s=0} = \left.\frac{s[u_2 e^{u_2 s} - u_1 e^{u_1 s}] - [e^{u_2 s} - e^{u_1 s}]}{(u_2 - u_2)s^2}\right|_{s=0}. \tag{2}$$

However evaluating this at $s = 0$ gives $\frac{0}{0}$. So instead we have to use the Hopital's rule

$$E[U] = \lim_{s\to 0} \frac{u_2 e^{u_2 s} - u_1 e^{u_1 s} + s[u_2^2 e^{u_2 s} - u_1^2 e^{u 1 s}] - [u_2 e^{u_2 s} - u_1 e^{u_1 s}]}{2(u_2 - u_1)} \tag{3}$$

$$= \frac{u_1 + u_2}{2} \tag{4}$$

To find the second moment we first find the second derivative of $\phi_U(s)$

$$E[U^2] = \frac{d^2\phi_U(s)}{ds^2} = \frac{s^2[u_2^2 e^{u_2 s} - a^2 e^{u_1 s}] - 2s[u_2 e^{u_2 s} - u_1 e^{u_1 s}] + 2[u_2 e^{u_2 s} - u_1 e_1^u s]}{(b-a)^3 s^3} \tag{5}$$

$$\tag{6}$$

Once again use the Hopital's Rule.

$$E[U^2] = \lim_{s\to 0} \frac{d^2\phi_U(s)}{ds^2} = lim \frac{s^2[u_2^3 e^{u_2 s} - u_1^3 e^{u_1 s}]}{3(u_2 - u_1)} = \frac{u2^3 - u1^3}{3(u_2 - u_1)} = \frac{(u_2^2 + u_1 u_2 + u_1^2)}{3} \tag{7}$$

Problem 3. Textbook problem 6.7.3 includes Matlab. Do it on your own rather than looking at the solution.

Solution 3:

a- The number of tests needed to identify 500 acceptable circuits is Pascal($k = 500, p = 0.8$), which has $E(L) = \frac{k}{P} = 625$ tests.

b-Let k denote the number of acceptable circuits in n=600 tests. Since k is binomial (n=600,p=0.8)

$$E[k] = np = 480 \tag{8}$$

$$Var[k] = np(1-p) = 96. \tag{9}$$

Using the central limit theorem we estimate

$$P(K \geq 500) = P(\frac{K - 500}{\sqrt{96}} \geq \frac{20}{\sqrt{96}}) = Q(\frac{20}{\sqrt{96}}) = 0.026 \tag{10}$$

c- Using MATLAB and the command

$$1 - \text{binomialcdf(600,0.8,499)} \tag{11}$$

we obtain 0.025.

d- We need to find the smallest value of n such that the binomial Random Variable K satisfies $P[k \geq 500] \geq 0.9$. Since $E[K] = np and Var[k] = np(1-p)$, the CLT is

$$P[K \geq 500] = P[\frac{K - np}{\sqrt{np(1-p)}} \geq \frac{500 - np}{\sqrt{np(1-p)}}] = 1 - \phi(z) = 0.9 = \phi(-z). \tag{12}$$

Then z=-1.29. Since p=0.8, we have that

$$np - 500 = 1.29\sqrt{np(1-p)} \tag{13}$$
$$n = 641.3 \tag{14}$$

Problem 4. Use the MGF to find the 1st, 2nd, 3rd and 4th moments of a Gaussian random variable with mean 0 and variance σ^2.
Solution 4: Using the moment generating function of X,

$$\phi_X(s) = e^{\frac{s^2\sigma^2}{2}}. \tag{15}$$

With this we can get the nth moments by taking the nth derivative with respect to s and setting s=0.

$$E(X) = \sigma^2 s e^{\frac{s^2\sigma^2}{2}}\Big|_{s=0} = 0 \tag{16}$$
$$E(X^2) = \sigma^2 e^{\frac{s^2\sigma^2}{2}} + \sigma^4 s^2 e^{\frac{s^2\sigma^2}{2}}\Big|_{s=0} = \sigma^2 \tag{17}$$
$$E(X^3) = (3\sigma^4 s + \sigma^6 s^3)e^{\sigma^2 s^2/2}\Big|_{s=0} = 0 \tag{18}$$
$$E(X^4) = (3\sigma^4 + 6\sigma^6 s^2 + \sigma^8 s^4)e^{\sigma^2 s^2/2}\Big|_{s=0} = 3\sigma^4 \tag{19}$$

As for $Y = X + \mu_Y$ so that Y is $\mathcal{N}(\mu, \sigma)$

$$E(Y^2) = E[(X+\mu)^2] = E[X^2 + \mu^2 + 2\mu X] = \sigma^2 + \mu^2. \tag{20}$$
$$E(Y^3) = E[(X+\mu)^3] = E[X^3 + \mu^3 + 3\mu X^2 + 3X\mu^2] = 3\mu\sigma^2 + \mu^3 \tag{21}$$
$$E(Y^4) = E[(X+\mu)^4] = E[X^4 + \mu^4 + 4\mu X^3 + 6\mu^2 X^2 + 4\mu^3 X] = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4 \tag{22}$$
$$\tag{23}$$

Problem 5. Textbook problem 6.6.2. Do it on your own rather than looking at the solution.
Solution 5: Knowing that the probability that the voice calls occurs is 0.8 and the probability that the data call occurs is 0.2 we define the random D_i as the number of data calls in a single

telephone call. It is obvious that for any i there are only two possible values for D_i, namely 0 and 1. Furthermore for all i the D_is are independent and identically distributed with the following pmf.

$$P_D[d] = \begin{pmatrix} 0.8 & d = 0 \\ 0.2 & d = 1 \end{pmatrix}$$

Then

$$E[D] = 0.2 \tag{24}$$
$$Var[D] = 0.16. \tag{25}$$

and then

a- $E(K_{100}) = 100E[D]$

b- $Var[K_{100}] = \sqrt{100Var[D]}$ =4

c- $P(K_{100} \geq 18) = 1 - \Phi\left(\frac{18-20}{4}\right) = 1 - \Phi(-0.5) = \Phi(0.5) = 0.6915.$

d- $P[16 \leq k_{100} \leq 24] = 1 - \Phi\left(\frac{24-20}{4}\right) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6826.$