ECE 341: Probability and Random Processes for Engineers, Spring 2012
Homework 10
Name:
Assigned: 03.28.2012
Due: 04.04.2012

Problem 1. Textbook problem 5.4.5. Do it on your own rather than looking at the solution.

Solution 1: Evaluate the joint density function, it is not equal to the product of marginal pdfs then they are not independent

$$f_{X1,X2,X3}(10,9,8) = 0 \neq f_{X1}(10)f_{X2}(9)f_{X3}(8) \tag{1}$$

Problem 2. Textbook problem 5.6.1. Do it on your own rather than looking at the solution.

Solution 2:

$$C_X = \begin{bmatrix} Var(X1) & Cov(X1,X2) \\ Cov(X1,X2) & Var(X2) \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \tag{2}$$

b- - Let $Y = [X1 \;\; X2]'$ since Y is linear combination of gaussian random variables then it is also a gaussian vector.

$$C_Y = AC_XA^T = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 28 & -66 \\ -66 & 252 \end{bmatrix} \tag{3}$$

Problem 3. Textbook problem 5.7.1. Do it on your own rather than looking at the solution.

Solution 3: The correlation matrix is

$$R_x = C_x - \mu_x\mu_x' = \begin{bmatrix} 20 & 30 & 25 \\ 30 & 68 & 46 \\ 25 & 46 & 40 \end{bmatrix} \tag{4}$$

b- Let $Y = [X1 \;\; X2]'$ since Y is subset of gaussian then it is also a gaussian vector

$$E(Y) = [E(X1) \;\; E(X2]' = [4 \;\; 8]' \tag{5}$$

$$C_Y = \begin{bmatrix} Var(X1) & Cov(X1,X2) \\ Cov(X1,X2) & Var(X2) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \tag{6}$$

$$\tag{7}$$

$$C_Y^{-1} = \frac{1}{20} \begin{bmatrix} 4 & +2 \\ +2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \tag{8}$$

$$\tag{9}$$

Now solving for the exponent $(Y-\mu_Y)K_Y^{-1}(Y-\mu_Y)$

$$\begin{bmatrix} y1-4 & y2-8\end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3}\end{bmatrix}\begin{bmatrix} y1-4 \\ y2-8\end{bmatrix} = \frac{y_1^2}{3}+\frac{y_1y_2}{3}-\frac{16y}{3}-\frac{20y_2}{3}+\frac{y_2^2}{3}+\frac{112}{3} \tag{10}$$

The PDF of Y is

$$f_Y(y) = \frac{1}{\sqrt{48\pi^2}}e^{(}-\frac{y_1^2}{3}+\frac{y_1y_2}{3}-\frac{16y}{3}-\frac{20y_2}{3}+\frac{y_2^2}{3}+\frac{112}{3}) \tag{11}$$

c- With the use of Q- function

$$P(X_1>8) = Q(2) = 0.0288 \tag{12}$$

Problem 4. Suppose X and Y are jointly Gaussian distributed with parameters $E[X]=\mu_x$, $E[Y]=\mu_y$, $Var(X)=\sigma_X^2$, $Var(Y)=\sigma_Y^2$ and $E[XY]=\rho_{XY}$. Are the following statements true or false?

1. X and Y are then also Gaussian with $X\sim\mathcal{N}(\mu_X,\sigma_X^2)$, and $Y\sim\mathcal{N}(\mu_Y,\sigma_y^2)$.
2. If $\rho_{XY}=0$ then X and Y are independent.
3. $Z=aX+bY$ for known constants a,b is Gaussian with mean $a\mu_X+b\mu_Y$.
4. $Z=aX+bY$ for known constants a,b is Gaussian with variance $a^2\sigma_X^2+b^2\sigma_Y^2$.
5. The linear minimum mean squared estimator of X in terms of Y is $\hat{X}=\mu_X+\rho_{XY}(Y-\mu_y)\frac{\sigma_X}{\sigma_Y}$.
6. The conditional distribution of $Y|X$ is Gaussian.
7. $E[X|Y]=\mu_X+\rho_{XY}(Y-\mu_y)\frac{\sigma_X}{\sigma_Y}$.

Solution 4:

1. X and Y are then also Gaussian with $X\sim\mathcal{N}(\mu_X,\sigma_X^2)$, and $Y\sim\mathcal{N}(\mu_Y,\sigma_y^2)$.
   Yes, this is true. A joint Gaussian distribution implies marginally distributed gaussian.
2. If $\rho_{XY}=0$ then X and Y are independent.
   ρ_{XY}= 0 then Cov(X,Y)=0 and E(XY) = E(X)E(Y) then X and Y are uncorrelated and in the case of gaussian only uncorrelated random variables implies independent.
3. $Z=aX+bY$ for known constants a,b is Gaussian with mean $a\mu_X+b\mu_Y$.
   This is true, $E(Z)=aE(X)$ + b$E(Y)$= $a\mu_X$+$b\mu_Y$
4. $Z=aX+bY$ for known constants a,b is Gaussian with variance $a^2\sigma_X^2+b^2\sigma_Y^2$.
   This is not true, since we did not account for the covariance between X and Y.
   Var(Z) = 2ab Cov(X,Y) + Var(X)+Var(Y)

5. The linear minimum mean squared estimator of X in terms of Y is $\hat{X} = \mu_X + \rho_{XY}(Y - \mu_y)\frac{\sigma_X}{\sigma_Y}$.
   True. This is the optimal estimator.

6. The conditional distribution of $Y|X$ is Gaussian.
   True, Conditioning preserves Gaussianity.

7. $E[X|Y] = \mu_X + \rho_{XY}(Y - \mu_y)\frac{\sigma_X}{\sigma_Y}$.
   True in fact this is the Linear minimum mean square estimate of X.

Problem 5. Suppose X and Y are zero-mean unit-variance jointly Gaussian random variables with correlation coefficient $\rho = 0.5$.

1. Find $Var(3X - 2Y)$.

2. Find the numerical value of $P[(3X - 2Y)^2 \leq 28]$.

3. Find the numerical value of $E[Y|X = 3]$.

Solution 5:

$$Var(3X - 2Y) = Cov(3X - 2Y, 3X - 2Y) = 9Var(X) + 4Var(Y) - 12Cov(XY) = 9 - 6 + 4 = 7 \tag{13}$$

Find the numerical value of $P[(3X - 2Y)^2 \leq 28]$.
Define a Random variable $Z = 3X - 2Y$. Keeping in mind that any linear combination of Gaussian is Gaussian then Z is a Gaussian with $E(Z) = 0$ and $Var(Z) = 0$

$$P(Z^2 \leq 28) = P(-\sqrt{28} \leq Z \leq \sqrt{28}) = \Phi(\frac{\sqrt{28}}{7}) - \Phi(\frac{-\sqrt{28}}{7}) = 2(\Phi(2) - 0.5) = 0.9545. \tag{14}$$

$$E[Y|X = 3] = \mu_Y + (x - \mu_x)\frac{Cov(XY)}{Var(X)} \tag{15}$$

$$= 0 + (3)\rho_{X,Y} = 1.5 \tag{16}$$