ECE 341: Probability and Random Processes for Engineers, Spring 2012
Homework 12
Name:
Assigned: 04.11.2012
Due: 04.18.2012

Problem 1. We model the noon-time temperature in Singapore (in degrees celsius) as X_n on day n, where X_n is a sequence of i.i.d. Gaussian random variables with a mean of 30 degrees celsius and standard deviation 5 degrees.

1. Consider the new random process $Y_k = \frac{X_{2k-1}+X_{2k}}{2}$ (the average temperature over two days. Is Y_k an i.i.d. random sequence?
2. Consider another new random process $W_k = \frac{X_n+X_{n-1}}{2}$ (moving average of temperatures over two days). Is W_n an i.i.d. random sequence?

Solution 1:
1. Each Y_k is the sum of two identical independent Gaussian random variables. Hence, each Y_k must have the same PDF. That is, the Y_k are identically distributed. Next, we observe that the sequence of Y_k is independent. To see this, we observe that each Y_k is composed of two samples of X_k that are unused by any other Y_j for $j \neq k$.

2. Each W_n is the sum of two identical independent Gaussian random variables. Hence, each W_n must have the same PDF. That is, the W_n are identically distributed. However, since W_{n-1} and W_n both use X_{n-1} in their averaging, W_{n-1} and W_n are dependent. We can verify this observation by calculating the covariance of W_{n-1} and W_n. First, we observe that for all n.

$$E[W_n] = \frac{E[X_n] + E[X_{n-1}]}{2} = 30 \tag{1}$$

Next we observe that W_n and W_{n-1} have covariance

$$\begin{aligned} Cov[W_{n-1}, W_n] =& E[W_{n-1}W_n] - E[W_{n-1}]E[W_n] && (2) \\ =& \frac{1}{4}E[(X_{n-1} + X_{n-2})(X_n + X_{n-1})] - 900 && (3) \end{aligned}$$

We observe that for n $\neq$ m, $E[X_nX_m] = E[X_n]E[X_m] = 900$ while

$$E[X_n^2] = Var[X_n] + (E[X_n])^2 = 916 \tag{4}$$

Thus,

$$Cov[W_{n-1}, W_n] = \frac{900 + 916 + 900 + 900}{4} - 900 = 4 \tag{5}$$

Since $Cov[W_{n-1}, W_n] \neq 0$, W_n and W_{n-1} must be dependent.

Problem 2. If X_n is an i.i.d. random sequence with mean $E[X_n] = \mu_x$ and variance $Var[X_n] = \sigma_x^2$, what is the auto-covariance $C_x[m,k]$?

Solution 2: The discrete time autocovariance function is

$$C_X[m,k] = E[(X_m - \mu_X)(X_{m+k} - \mu_X)] \tag{6}$$

for $k = 0, C_X[m,0] = Var[X_m] = \sigma_X^2$ For $k \neq 0$, X_m and X_{m+k} are independent so that

$$C_X[m,k] = E[(X_m - \mu_X)]E[(X_{m+k} - \mu_X)] = 0 \tag{7}$$

Thus the autocovariance of X_n is

$$C_X[k,m] = \begin{pmatrix} 0 & k \neq 0 \\ \sigma_X^2 & k = 0 \end{pmatrix}$$

Problem 3. Let $X(t)$ be a stationary process (strictly stationary). Consider $Y(t) = X(t+a)$; is $Y(t)$ also stationary or not?

Solution 3:

For an arbitrary set of samples $Y(t_1), ..., Y(t_k)$, we observe that $Y(t_j) = X(t_j + a)$. This implies

$f_{Y(t_1),...,Y(t_k)}(y_1, ..., y_k) = f_{X(t_1+a),...,X(t_k+a)}(y_1, ..., y_k)$
$f_{Y(t_1+\tau),...,Y(t_k+\tau)}(y_1, ..., y_k) = f_{X(t_1+\tau+a),...,X(t_k+\tau+a)}(y_1, ..., yk)$
Since X(t) is a stationary process,
$f_{X(t_1+\tau+a),...,X(t_k+\tau+a)}(y_1, ..., y_k) = f_{X(t_1+a),...,X(t_k+a)}(y_1, ..., y_k)$
$f_{Y(t_1+\tau),...,Y(t_k+\tau)}(y_1, ..., y_k) = f_{X(t_1+a),...,X(t_k+a)}(y_1, ..., y_k)$
$= f_{Y(t_1),...,Y(t_k)}(y_1, ..., y_k)$

We can conclude that Y(t) is a stationary process.

Problem 4. Similar to what we showed in class (but now you need to work it out), let $X(t)$ be a wide-sense stationary random process with average power equal to 1. Let Θ denote a random variable with uniform distribution over $[0, 2\pi]$, and let Θ and $X(t)$ be independent.

1. Find $E[X^2(t)]$.
2. Find $E[\cos(2\pi f_c t + \Theta]$ (show it, don't just quote class).
3. Let $Y(t) = X(t)\cos(2\pi f_c t + \Theta)$. What is $E[Y(t)]$?

4. What is the average power of $Y(t)$?

Solution 4:
1. In the problem statement, we are told that$X(t)$ has average power equal to 1, the average power of $X(t)$ is $E[X^2(t)] = 1$.

2. Since Θ has a uniform PDF over $[0, 2\pi]$

$$f_\Theta[\theta] = \begin{pmatrix} \frac{1}{2\pi} & 0 \leq \theta \leq 2\pi \\ 0 & \text{otherwise} \end{pmatrix}$$

The expected value of the random phase cosine is

$$\begin{aligned} E[cos(2\pi f_c t + \Theta)] &= \int_{-\infty}^{\infty} cos(2\pi f_c t + \Theta) f_\Theta[\theta] d\theta && (8) \\ &= \int_{0}^{2\pi} cos(2\pi f_c t + \Theta) \frac{1}{2\pi} d\theta && (9) \\ &= 0 && (10) \end{aligned}$$

3. Since $X(t)$ and Θ are independent

$$E[Y(t)] = E[X(t) cos(2\pi f_c t + \Theta)] = E[X(t)] E[cos(2\pi f_c t + \Theta)] = 0 \quad (11)$$

Note that the mean of $Y(t)$ is zero no matter what the mean of $X(t)$ since the random phase cosine has zero mean.

4.Independence of $X(t)$ and Θ results in the average power of $Y(t)$ being

$$\begin{aligned} E[Y^2(t)] =& E[X^2(t) cos^2(2\pi f_c t + \Theta)] && (12) \\ &= E[X^2(t)] E[cos^2(2\pi f_c t + \Theta)] && (13) \\ &= E[cos^2(2\pi f_c t + \Theta)] && (14) \end{aligned}$$

Note that we have used the fact from part (a) that X(t) has unity average power. To finish the problem, we use the trigonometric identity $cos^2 a = \frac{cos 2a + 1}{2}$. This yields

$$E[Y^2(t)] = E[\frac{1}{2}(1 + cos(2\pi(2f_c)t + \Theta] = \frac{1}{2} \quad (15)$$

Note that $E[cos(2\pi(2f_c)t + \Theta)] = 0$ by the argument given in part 2 with $2f_c$ replacing fc.