Review for midterm 1

1. Given that 12 airplanes crash in a year, and that the probability that any given crash is equally likely to occur in any month, what is the probability that there is exactly 1 crash each month?

ways of distributing 12 plane crashes over 12 moths? 12^{12}.

ways ~~for~~ to have 1 plane crash per month? $12!$

$$\frac{12!}{12^{12}} = 5.37 \times 10^{-5}$$

---

2. A batch of 50 items contains 10 defective items. Suppose 10 items are selected at random and tested. What is the prob. that exactly 5 of the items tested are defective?

possible ways of selecting 10 out of 50 items: $\binom{50}{10}$

ways of selecting exactly 5 defective + 5 non-defective items out of the 50 items?

$$\underbrace{\binom{10}{5}}_{5 \text{ defective}} \underbrace{\binom{40}{5}}_{5 \text{ non-defetive}}$$

$\therefore$ Prob $= \dfrac{\binom{10}{5}\binom{40}{5}}{\binom{50}{10}}$

3 A production line yields two types of devices. Type 1 devices occur with prob. α and work for a relatively short time that is geometrically distributed with parameter r. Type 2 devices work much longer, occur with prob. $1-\alpha$ and have a lifetime that is geometrically distributed with parameter s. Let X be the lifetime of a random device. Find the p.m.f. of X. 2

$S_X = \{1, 2, \ldots \}$

$$P_X(x) = P_{X|B_1}(x)P[B_1] + P_{X|B_2}(x)P[B_2].$$

this is true as $\{B_1, B_2\}$ form an event space.

where event B_1 = pick device of type 1, $\therefore P[B_1] = \alpha$

event B_2 = pick device of type 2 $\therefore P[B_2] = 1-\alpha$.

$$P_{X|B_1}(x) = (1-r)^{x-1} r \qquad x = 1, 2, \ldots$$

$$P_{X|B_2}(x) = (1-s)^{x-1} s \qquad x = 1, 2, \ldots$$

---

4 The # N of queries arriving in t seconds at a call center is a Poisson R.V. with $\alpha = \lambda t$, where λ is the average arrival rate in queries/second. Assume that the arrival rate is $\underbrace{\text{four queries per minute}}_{\lambda = \frac{1}{15} \text{ queries/sec}}$.

- Find Prob. of >4 queries in 10 seconds.

$$P[N>4] = 1 - P[N \le 4]$$
$$= 1 - \sum_{k=0}^{4} \frac{(2/3)^k e^{-2/3}}{k!} \qquad \left(\frac{2}{3} = \alpha = \frac{1}{15} \times 10\right)$$

$$\boxed{P(k) = \frac{(\lambda T)^k e^{-\lambda T}}{k!}}$$

Poisson prob k arrivals in T seconds with λ arrival rate.

$k = 0, 1, 2, \ldots$

- Find the prob. of <5 queries in 2 minutes.

$$P[N<5] = \sum_{k=0}^{4} \frac{(8)^k e^{-8}}{k!} \qquad \left(8 = \alpha = \frac{1}{15} \times 120\right)$$

$$P(k) = \frac{\alpha^k e^{-\alpha}}{k!}$$

$(\alpha = \lambda T)$

5 Find $E[X^2]$ of a binomial random variable with paramaters n (# Bernouilli trials) and p (prob. of success).

X is binomial, Then $P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$ $\quad k=0,1,\ldots,n.$

$$E[X^2] = \sum_{k=0}^{n} k^2 P_X(k) = \sum_{k=0}^{n} k^2 \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}.$$

$$= \sum_{k=1}^{n} k \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} k \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k'=0}^{n-1} (k'+1) \frac{(n-1)!}{(k')!(n-k'-1)!} p^{k'} (1-p)^{n-1-k'} \qquad k' = k-1$$

$$= np \sum_{k'=0}^{n-1} (k'+1) \binom{n-1}{k'} p^{k'} (1-p)^{(n-1)-k'}$$

$$= np \left\{ \sum_{k'=0}^{n-1} k' \binom{n-1}{k'} p^{k'} (1-p)^{(n-1)-k'} + \sum_{k'=0}^{n-1} \binom{n-1}{k'} p^{k'} (1-p)^{(n-1-k')} \right.$$

$$= np \Big\{ \underbrace{E[\text{Binomial}(n-1,p)]}_{(n-1)p.} + \underbrace{\sum \text{Binomial}(n-1,p)}_{1} \Big].$$

mean of binomial with parameters $(n-1), p.$

$$= np\{(n-1)p + 1\}.$$