ECE 341 Probability and Random Processes for Engineers - MIDTERM 2

03/16/2012, DH 210.

- This exam has 10 questions each is worth 10 points.
- You will be given the full 50 minutes. **Use it wisely!** Many of the problems have short answers; try to find shortcuts. Do questions that you think you can answer correctly first.
- You may bring and use two 8.5x11" double-sided crib sheet.
- No other notes or books are permitted.
- No calculators are permitted.
- Talking, passing notes, copying (and all other forms of cheating) is forbidden.
- Make sure you explain your answers in a way that illustrates your understanding of the problem. Ideas are important, not just the calculation.
- Partial marks will be given.
- Write all answers directly on this exam.

Your name: Solutions

Your UIN: ______________________

Your signature: ______________________

The exam has 6 questions, for a total of 100 points.

Question:	1	2	3	4	5	6	Total
Points:	10	20	20	15	20	15	100
Score:							

1. (10 points) The transmission of digital data over a noisy communication channel is modeled as adding zero-mean, Gaussian noise N (of variance σ_N^2) to a discrete random variable X, where X takes on values $+5$ and -5 with equal probability. The detected signal is $Y = X + N$ (i.e. the received only gets to see Y). If $Y > 0$ when $X = -5$ or when $Y < 0$ when $X = +5$ a transmission error has occurred. Find the channel SNR (in dB) and noise variance σ_N^2 that result in a probability of error of 3.9×10^{-4} (no calculators needed, just write an expression with as many of the numbers filled in as possible).

From class, recall that

$$\text{Prob}[\text{error}] = Q(\sqrt{SNR})$$

If we want $\text{Prob}[\text{error}] = Q(\sqrt{SNR}) = 3.9 \times 10^{-4}$

then using the $Q(\cdot)$ function table, we see that

$$\sqrt{SNR} = 3.36 \Rightarrow SNR = (3.36)^2$$

$$\Rightarrow 10\log_{10}((3.36)^2) = 20\log_{10}(3.36) \text{ (dB)}$$

Channel SNR needed.

Since $SNR = \dfrac{E[X^2]}{E[N^2]} = \dfrac{E[X^2]}{var(N)} = \dfrac{25}{\sigma_N^2} = (3.36)^2$

$$\Rightarrow \sigma_N^2 = \frac{25}{(3.36)^2}.$$

Noise variance needed.

2. (10 points) The discrete random variables X and Y are described by the following joint PMF

	$Y=-1$	$Y=0$	$Y=1$	$Y=2$
$X=0$	0	1/4	0	1/4
$X=2$	1/4	0	1/4	0

(a) (5 points) Are X and Y independent?

(b) (5 points) Find $E[X+3Y]$.

(a) $P_X(0) = 1/2$, $P_X(2) = 1/2$

$P_Y(-1) = 1/4$, $P_Y(0) = 1/4$, $P_Y(1) = 1/4$, $P_Y(2) = 1/4$.

Not independent as $\left(P_{XY}(0,-1) = 0\right) \neq \left(P_X(0)P_Y(-1) = 1/8\right)$

(b) $E[X+3Y] = E[X] + 3E[Y]$

$= \frac{1}{2}\cdot 0 + \frac{1}{2}(2) + 3\left[\frac{1}{4}(-1) + \frac{1}{4}(0) + \frac{1}{4}(1) + \frac{1}{4}(2)\right]$

$= 1 + 3\left[\frac{1}{2}\right] = \frac{5}{2}.$

3. We are given that two random variables X and Y are jointly Gaussian, with the following pdf:

$$f_{X,Y}(x,y) = \frac{1}{2\pi(2)(4)(0.8)} \exp\left[-\frac{\left(\frac{x-1}{2}\right)^2 + \frac{2(0.6)(x-1)(y+1)}{(2)(4)} + \left(\frac{y+1}{4}\right)^2}{1.28}\right]$$

(a) (10 points) Find the linear minimum mean squared error estimator of X based on Y (given Y).

(b) (10 points) The actual optimal non-linear estimator of X given Y is given by $\hat{X}_{opt} = E[X|Y]$. Find $\hat{X}_{opt}$.

First, we recognize + read off the following from the above pdf:

$\mu_X = 1$, $\sigma_X = 2$, $\mu_Y = -1$, $\sigma_Y = 4$, $\rho_{XY} = 0.6$

(a) From class, we recall that the linear MMSE estimator of X in terms of Y is:

$$\hat{X} = \rho_{XY}(Y-\mu_Y)\left(\frac{\sigma_X}{\sigma_Y}\right) + \mu_X$$

$$= 0.6(Y+1)\left(\frac{2}{4}\right) + 1 = 0.3(Y+1) + 1$$

(b) $\hat{X}_{opt} = E[X|Y]$

Since X, Y are jointly Gaussian we know $X|Y$ is also Gaussian and we recall ~~that~~ its ~~...~~ pdf:

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi\tilde{\sigma}_X^2}} e^{-\frac{(x-\tilde{\mu}_X)^2}{2\tilde{\sigma}_X^2}} \quad \text{where} \quad \tilde{\mu}_X = \mu_X + \frac{\rho_{XY}\sigma_X}{\sigma_Y}(y-\mu_Y)$$

$$\tilde{\sigma}_X^2 = \sigma_X^2(1-\rho_{XY}^2)$$

So, $\hat{X}_{opt} = E[X|Y]$ is the mean of this distribution

$$= \mu_X + \frac{\rho_{XY}\sigma_X}{\sigma_Y}(Y-\mu_Y)$$

$$= 1 + 0.6\left(\frac{2}{4}\right)(Y+1) = \text{same as above!}$$

Points earned: __________ out of a possible 20 points

4. The joint pdf of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} ce^{-x}e^{-y} & 0 \le y \le x < \infty \\ 0 & \text{else} \end{cases}$$

(a) (5 points) Find the constant c.

(b) (5 points) Are X and Y independent?

(c) (5 points) Find $P[X + Y \le 1]$.

(a) $$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy \overset{\text{WANT}}{=} 1 = \int_0^\infty\int_0^x ce^{-x}e^{-y}\,dy\,dx = \int_0^\infty ce^{-x}\left(-e^{-y}\right)\Big|_0^x\,dx$$

$$= \int_0^\infty ce^{-x}\left(-e^{-x}+1\right)dx = c\left[\frac{1}{2}e^{-2x} - e^{-x}\right]_0^\infty = c\left[\frac{1}{2}\right] = 1$$

$$\Rightarrow \boxed{c = 2}$$

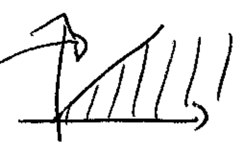
left out.

(b) ~~$f_X(x) =$~~ No, and we don't even have to calculate.

$f_X(x)$ is defined on $0 \le x \le \infty$ } and will be non-zero there (can easily see, or just calculate $f_X(x)$ or $f_Y(y)$)

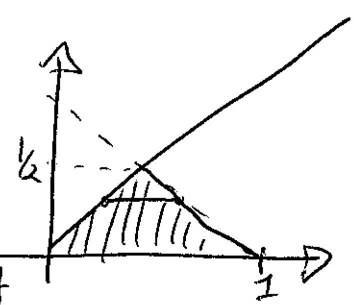
$f_Y(y)$ is defined on $0 \le y \le \infty$

But, $f_{XY}(x,y)$ is non-zero only on



so can find examples of points in where ~~not independ~~ $f_{XY}(x,y) = 0 \ne f_X(x)f_Y(y)$.

(c) $$P[X+Y \le 1] = \int_{y=0}^{1/2}\int_{x=y}^{1-y} 2e^{-x}e^{-y}\,dx\,dy$$



$$= \int_0^{1/2} 2e^{-y}\left[-e^{-x}\right]_y^{1-y}dy = \int_0^{1/2} 2e^{-y}\left[e^{-y} - e^{-(1-y)}\right]dy$$

$$= -e^{-2y} - 2e^{-1}y\Big|_0^{1/2} = -e^{-1} - e^{-1} + 1 = 1 - 2e^{-1}$$

Points earned: ______________ out of a possible 15 points

5. (20 points) X is a continuous random variable selected at random from $[0,1]$; Y is then selected at random from the interval $(0,X)$. Find the CDF of Y.

When $X=x$, Y is uniform on $[0,x]$ so the conditional cdf given $X=x$ is

$$F_Y(y|x) = P[Y \le y | X=x]$$

$$= \begin{cases} \frac{y}{x} & 0 \le y \le x \\ 1 & x < y. \end{cases}$$

(from uniform on $[0,x]$ the pdf is $\frac{1}{x}$, so cdf is $\frac{y}{x}$).

Then $F_Y(y) = P[Y \le y] = \int_{x=0}^{1} P[Y \le y | X=x] f_X(x)\, dx.$

$$= \int_{x=0}^{y} 1 \cdot dx + \int_{x=y}^{1} \frac{y}{x}\, dx = y + y \ln x \Big|_{x=y}^{1}$$

$$= y + \underbrace{y \ln 1}_{0} - y \ln y = y - y \ln y.$$

Points earned: ____________ out of a possible 20 points

6. True or False (T/F) and short answers (SA).
   (a) (3 points) (T/F) If the correlation coefficient between X and Y is zero, they are independent.
   (b) (3 points) (T/F) If X and Y are jointly Gaussian then the marginals of X and Y are also Gaussian.
   (c) (3 points) (T/F) Flip a coin. Let X be the number of heads and Y be the number of tails. X and Y are independent.
   (d) (3 points) (SA) We repeatedly transmit a packet, which is correctly received with probability p, until it is correctly received. Find the expected number of transmissions needed for the packet to be correctly received.
   (e) (3 points) (SA) What is the relationship between the CDF and PDF of a continuous random variable? Write the mathematical expressions.

(a) False. Independent implies $\rho_{XY}=0$ but $\rho_{XY}=0$ does NOT imply independence.

(b). True. shown in class.

(c) False. If we know X then we know Y, They are definitely NOT independent.

(d) We recognize this as a geometric random variable with parameter p which has mean $\frac{1}{p}$,

(e) PDF of a continuous RV X is $f_X(x)$, CDF is $F_X(x)$ related as

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du.$$

Points earned: ______ out of a possible 15 points