ECE 341 Probability and Random Processes for Engineers - MIDTERM 1

02/06/2012, DH 210.

- This exam has 5 questions, each of which is worth 20 points.
- You will be given the full 50 minutes. **Use it wisely!** Many of the problems have short answers; try to find shortcuts. Do questions that you think you can answer correctly first.
- You may bring and use one 8.5x11" double-sided crib sheet.
- No other notes or books are permitted.
- No calculators are permitted.
- Talking, passing notes, copying (and all other forms of cheating) is forbidden.
- Make sure you explain your answers in a way that illustrates your understanding of the problem. Ideas are important, not just the calculation.
- Partial marks will be given.
- Write all answers directly on this exam.

Your name: Solutions

Your UIN: __________

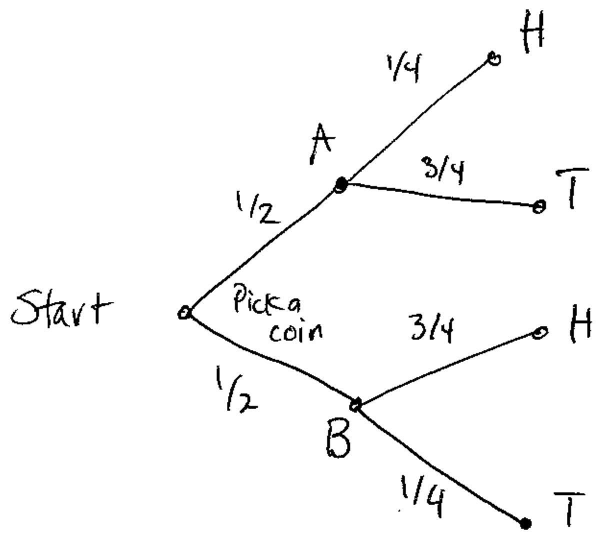
Your signature: __________

The exam has 5 questions, for a total of 100 points.

Question:	1	2	3	4	5	Total
Points:	15	15	30	20	20	100
Score:						

1. (15 points) *Biased coins.* You have two biased coins. Coin A comes up heads with probability 1/4 and coin B comes up heads with probability 3/4. You are not sure which coin is which, so you choose one coin at random and flip it. If the flip is heads, you guess that the flipped coin is B; otherwise you guess that the flipped coin is A. What is the probability that your guess is correct?

Make a tree diagram



→ outcome pick A and see H ⇒ prob. 1/8

→ outcome pick A and see T ⇒ prob. 3/8

→ outcome pick B and see H ⇒ prob. 3/8

→ outcome pick B and see T ⇒ prob. 1/8.

We ~~make a mistake~~ are correct on outcomes BH and AT.

So, $$\text{Prob}[\text{correct}] = P[BH \cup AT]$$
$$= P[BH] + P[AT]$$
$$= 3/8 + 3/8$$
$$= 6/8 = 3/4.$$

Points earned: ____________ out of a possible 15 points

2. (15 points) *Variance.* Find the variance of $Y = aX + b$ in terms of a, b, μ_x (mean of X), σ_x (standard deviation of X), $Var[X]$ (variance of X), or a subset thereof.

$$Var[aX+b] = E[(aX+b - E[aX+b])^2]$$

(Note that $E[aX+b] = aE[X]+b = a\mu_x + b$. (let $\mu_x = E[X]$))

$$= E[(aX+b - a\mu_x - b)^2]$$

$$= E[(a(X-\mu_x))^2]$$

$$= E[a^2(X-\mu_x)^2]$$

$$= a^2 E[(X-\mu_x)^2]$$

$$= a^2 Var[X].$$

Points earned: ____________ out of a possible 15 points

3. *ACKs and NAKs.* A source wishes to transmit data packets to a receiver over a radio link. The receiver uses error detection (like a CRC check) to identify packets that have been corrupted. When a packet is received error-free, the receiver sends an acknowledgement (ACK) back to the source. When the receiver gets a packet with errors, a negative acknowledgement (NAK) message is sent to the source. Each time the source receives a NAK, the packet is re-transmitted. We assume that each packet transmission is independently corrupted by errors with probability q.
   (a) (15 points) Find the p.m.f. of X, the number of times that a packet is transmitted by the source.
   (b) (15 points) Suppose each packet takes 1 millisecond to transmit and that the source waits an additional millisecond to receive the acknowledgement (ACK or NAK) before retransmitting. Let T equal the time required until the packet is successfully received. What is p.m.f. of T?

(a) Source re-transmits until correctly received. This is a geometric random variable with prob(success) $= 1-q$.

$$P_X(x) = \begin{cases} q^{x-1}(1-q) & x = 1, 2, \ldots \\ 0 & \text{else} \end{cases}$$

to be correct at x-th transmission need $x-1$ failures.

(b) If X transmissions are needed to correctly receive a packet then we need $T = 2X - 1$ milliseconds (each failure is $2X$). Therefore, the range of T is $S_T = \{1, 3, 5, \ldots\}$ and the p.m.f. may be directly obtained from the p.m.f. for X as

$$P_T(t) = P_X\left(\underbrace{(t+1)/2}_{x = \frac{t+1}{2}}\right) = \begin{cases} q^{\frac{(t-1)}{2}}(1-q) & t = 1, 3, 5, \ldots \\ 0 & \text{else.} \end{cases}$$

4. (20 points) *Radars.* Radars detect flying objects by measuring the power reflected from them. The reflected power of an aircraft can be modeled as a random variable Y with p.d.f.

$$f_Y(y) = \frac{1}{P_0} e^{-y/P_0}, \ \ y \geq 0$$

where P_0 is some constant. The aircraft is correctly identified by the radar if the reflected power of the aircraft is larger than its average value. What is the probability that an aircraft is correctly identified?

The reflected power Y follows an exponential distribution. with $\lambda = 1/P_0$. Thus, we know $E[Y] = \frac{1}{\lambda} = P_0$.

The probability that the aircraft is correctly identified is

$$P[Y > P_0] = \int_{P_0}^{\infty} \frac{1}{P_0} e^{-y/P_0} dy = -e^{-y/P_0} \Big|_{P_0}^{\infty} = \frac{1}{e}.$$

Points earned: ______________ out of a possible 20 points

5. *Words.* Consider a language containing four letters: A,B,C,D.
   (a) (~~10~~ ~~6~~ 7 points) How many 3 letter words can you form in this language?
   (b) (~~10~~ ~~6~~ 7 points) How many four-letter words can you form if each letter appears only once in each word?
   (c) (~~8~~ 6 points) How many orderings of the letters ABBA are there, if we don't distinguish between the two A's nor between the two B's?

(a) $\left.\begin{array}{l}\text{1st letter 4 choices} \\ \times \text{ 2nd letter 4 choices} \\ \times \text{ 3rd letter 4 choices}\end{array}\right\}$ 4^3 possible words

(b) $4!$ since 4 choices for 1^{st} letter, 3 for 2^{nd}, 2 for 3^{rd}, 1 for last.

(c) AABB
ABBA
BAAB
BABA
BBAA
ABAB.

6 orderings.

6. *BONUS QUESTION - attempt only if have time and enjoy it!*

Suppose a fair die is repeatedly rolled until each of the numbers one through six shows at least once. What is the mean number of rolls?

The total # rolls, R, can be expressed as $R = R_1 + R_2 + \cdots + R_6$ where R_i = # of rolls made after $i-1$ distinct numbers have shown up, up to and including the roll such that the i-th distinct # shows.

E.g. if the sequence of numbers rolled is

2 4 2 3 4 4 3 5 3 5 4 4 6 2 3 3 4 1

put a bar just after each roll that shows a new distinct # to get

2 | 4 | 2 3 | 4 4 3 5 | 3 5 4 4 6 | 2 3 3 4 1 |

$R_1 = 1,\ R_2 = 1,\ R_3 = 2,\ R_4 = 4,\ R_5 = 5,\ R_6 = 5$

After $i-1$ numbers have shown, the probability each subsequent roll is distinct from those $i-1$ numbers is $\frac{6-i+1}{6}$.

Thus, R_i has a geometric distribution with parameter $\frac{6-i+1}{6}$.

Thus, $E[R_i] = \frac{6}{6-i+1}$.

Hence,
$$E[R] = E[R_1 + R_2 + \cdots + R_6]$$
$$= E[R_1] + E[R_2] + \cdots + E[R_6]$$
$$= \frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = 6\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right).$$