Lattice Coding for the Two-way Line Network

Yiwei Song, Natasha Devroye, Huai-Rong Shao and Chiu Ngo

***Abstract*—Full-duplex allows for the simultaneous flow of information in two directions, and in point-to-point Gaussian two-way channels, doubles capacity. A two-way line network where two sources exchange messages through multiple serial relays is considered. It is shown that when all nodes are full duplex, one may achieve to within a constant gap, independent of the number of relays, of capacity of two parallel one-way line networks. This shows that, even in the presence of relays which carry information in two directions, full duplex is able to approximately double capacity. A novel lattice coding scheme is developed for the two-way line network with two relays which may be extended to an arbitrary number of relays, and to half-duplex scenarios. The key technical contribution is the achievability strategy, where each relay decodes the sum of several signals (using lattice codes) and then re-encodes it into another lattice codeword. This allows other nodes to again decode sums of codewords. The presented lattice-coding-based scheme ensures that both directions simultaneously fully utilize the relays' powers, even for asymmetric channels. The symmetric rate achieved by the proposed scheme is within** $0.5\log 5$ **bit/Hz/s of the symmetric rate capacity regardless of the number of relays.**

I. Introduction

The capacity region of a full-duplex point-to-point two-way Gaussian channel $1 \leftrightarrow 2$[1], where two nodes simultaneously transmit and receive signals in order to exchange two independent messages, is known to be equal to the capacity of two parallel or simultaneously operating one-way links. The ability of the nodes to transmit and receive at the same time, or operate under full duplex conditions, doubles the capacity of this network over its half-duplex counterpart – where a node may either transmit or receive at any given time, but not both – in which users time-share the channel [2]. This may be viewed as a canonical example of how full-duplex "doubles" capacity.

In recent years the usage of relays, or message-less nodes which seek to aid in the communication of other nodes' messages, is becoming more and more prevalent. Relays, like sources and destinations may again operate under half or full duplex constraints. An extension of the canonical two-way Gaussian channel to incorporate a relay node is the two-way Gaussian relay channel model $1 \leftrightarrow 2 \leftrightarrow 3$, in which nodes 1 and 3 exchange messages with the help of a relay node 2. For full-duplex nodes, a lattice-coding based scheme has been shown to achieve within 0.5 bit/sec/Hz per user of capacity [2], where the outer bound used is that of two parallel channels $1 \rightarrow 2 \rightarrow 3$ and $1 \leftarrow 2 \leftarrow 3$. That is, through the use of lattice codes we can "almost" double the network capacity of this two-way relay channel, relative to half duplex, time-sharing based schemes. The reason lattice codes, but not random codes, are able to do so is because of their linearity property – the sum of two lattice codewords is still a codeword. If the two users employ properly chosen lattice codewords, the relay node may decode the sum of the codewords from both users directly at higher rates than decoding them individually (which would be needed in a i.i.d. random coding based decode-and-forward scheme). It is then sufficient for the relay to broadcast this sum of codewords to both users since each user, knowing the sum and its own message, may determine the other desired message.

Past work. The capacity region of the point-to-point Gaussian full-duplex two-way channel was shown to be equal to two parallel one-way Gaussian channels in [3]. The point-to-point two-way channel was extended to a two-way relay channel in, among many others, [4]–[6], and in particular [2], [7]–[9] where lattice codes were shown in [2] to achieve within 0.5 bit/sec/Hz of the capacity. Lattice codes have been used to derive achievable rates for multi-user networks beyond the two-way relay channel, often outperforming i.i.d. random codes in certain scenarios, particularly when interested in decoding a linear combination of the received codewords rather than the individual codewords. Nested lattice codes have been shown to be capacity achieving in the point-to-point Gaussian channel [10], the Gaussian Multiple-access Channel [8], Broadcast Channel [11], and to achieve the same rates as those achieved by i.i.d. Gaussian codes in the Decode-and-Forward rate [7], [12] and Compress-and-Forward rates [7] of the Relay Channel [13]. Lattice codes may further be used in achieving the capacity of Gaussian channels with interference or state known at the transmitter (but not receiver) [14] using a lattice equivalent [11] of dirty-paper coding (DPC) [15]. The nested lattice approach of [11] for the dirty-paper channel is extended to dirty-paper networks in [16]. Lattice codes have also been shown to approximately achieve the capacity of the symmetric K-user interference channel [17], using the Compute-and-Forward framework for decoding linear equations of codewords of [8].

This work. In this work, we are interested in determining whether, when we add an arbitrary number of relays to form a two-way full duplex Gaussian line network, we can still "almost" achieve the capacity of two one-way multi-relay or

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[1]We use the notation $1 \leftrightarrow 2$ to informally denote that there exists a two-way Gaussian noise channel between nodes 1 and 2 with channel outputs Y_1, Y_2 related to channel inputs X_1, X_2 and independent additive Gaussian noise Z_1, Z_2 of arbitrary variance as $Y_1 = X_2 + Z_1$, $Y_2 = X_1 + Z_2$ at each channel use, subject to transmit power constraints.

[2]For additive white Gaussian noise channels subject to duty cycle and power constraints, creative on-off patterns may achieve better rates than naive time-sharing [1].

line networks operating in parallel *(i.e. full-duplex would still "double" the capacity, even in a line network).* We answer this in the positive. We first consider the two-way two-relay line network: $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$ where two user Nodes 1 and 4 exchange information with each other through the relay nodes 2 and 3. This is related to the work of [18], which considers the throughput of i.i.d. random code-based Amplify-and-Forward and Decode-and-Forward approaches for this channel model, or the i.i.d. random coding based schemes of [19] and [20] where there are links between all nodes rather than just neighboring nodes. This model is also different from that in [21] where a two-way relay channel with two *parallel* (rather than serial) relays are considered.

Contributions. The two-way two-relay channel $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$ is a generalization of the two-way relay channel $1 \leftrightarrow 2 \leftrightarrow 3$ to multiple relays. The key difficulty lies in having the relays successfully handle the data in both directions simultaneously. We propose to do so, for the first time, using a lattice-based scheme where all nodes transmit lattice codewords and each relay node decodes a sum of these codewords. This scheme may be seen as a generalization of the lattice-based scheme of [2], [9] for the two-way relay channel. However, this generalization is *not* straightforward due to the presence of multiple relays and hence the need to repeatedly be able to decode the sum of codewords. One way to enable this is to have the relays employ lattice codewords as well – something not required in the two-way relay channel. We thus present a new lattice-based broadcast scheme which is able to work for asymmetric links to the relay, i.e. in full generality, which is not immediate.[3] Another way in which the extension to multiple relays is not a straightforward extension of the two-way relay channel lattice-based achievability scheme is that our scheme consists of multiple block Markov phases, whereas the single relay scheme only consists of two phases. In each of our multiple block Markov phases, the end users send new messages encoded by lattice codewords and the relays decode a combination of lattice codewords. The relays first fold (modulo back) the decoded lattice codeword combinations, scale the result by the transmit power, and then broadcast the resulting lattice codewords. The novelty lies in the specially constructed lattice codes which ensure that the "folding back" operation is information lossless, with the appropriate side information. The folding enables the messages for both directions to fully utilize the relay transmit power, even under asymmetric channel conditions to the relay. Furthermore, all decoders are lattice decoders (more computationally efficient than joint typicality decoders) and only a single nested lattice codebook pair is needed. Finally, this scheme is extended to the general multi-relay line network: $1 \leftrightarrow 2 \leftrightarrow \cdots \leftrightarrow D-1 \leftrightarrow D$. We are still able to demonstrate a constant gap from the symmetric capacity, and interestingly, this gap is independent of the number of nodes.

Outline. We first outline some definitions, notation, and technical lemmas for nested lattice codes in Section II. We outline the channel model in Section III. We then outline a lattice-based strategy for the broadcast phase of the two-way (single) relay channel with asymmetric uplinks in Section IV, which includes the key technical novelty- specially constructed lattice codes and the folding operation which allows the relays to intuitively spread the signals traveling in both directions to utilize the relay's entire transmit power. We present the main achievable rate regions for the two-way two-relay channel in Section V before extending both our scheme and constant gap result to an arbitrary number of relays in Section VI, where the gap turns out to be independent of the number of relays. In Section VII we remark that our achievability scheme may be readily extended to half-duplex nodes as well.

[3]Being able to broadcast using lattice codes is otherwise obvious when the powers, noises and channels are all symmetric in the two links to the relay.

II. Preliminaries on lattice codes and notation

We now define our notation for lattice codes in Subsection II-A, define properties of nested lattice codes in Subsection II-B and technical lemmas in Subsection II-C.

A. Lattice codes

Our notation for (nested) lattice codes for transmission over AWGN channels follows that of [11], [22]; comprehensive treatments may be found in [10], [11], [23] and in particular [24]. An n-dimensional lattice Λ is a discrete subgroup of Euclidean space $\mathbb{R}^n$ with Euclidean norm $||\cdot||$ under vector addition and may be expressed as all integral combinations of basis vectors $\mathbf{b_i} \in \mathbb{R}^n$

$$\Lambda = \{\lambda = \mathbf{B}\ \mathbf{i} :\ \mathbf{i} \in \mathbb{Z}^n\},$$

for $\mathbb{Z}$ the set of integers, $n \in \mathbb{Z}_+$, and $\mathbf{B} := [\mathbf{b_1}|\mathbf{b_2}|\cdots\mathbf{b_n}]$ the $n\times n$ generator matrix corresponding to the lattice Λ. We use bold $\mathbf{x}$ to denote column vectors, $\mathbf{x}^T$ to denote the transpose of the vector $\mathbf{x}$. All vectors lie in $\mathbb{R}^n$ unless otherwise stated, and all logarithms are base 2. Let $\mathbf{0}$ denote the all zeros vector of length n, $\mathbf{I}$ denote the $n\times n$ identity matrix, and $\mathcal{N}(\mu, \sigma^2)$ denote a Gaussian distribution with mean vector μ and covariance matrix σ^2. Let $|\mathcal{C}|$ denote the cardinality of the set $\mathcal{C}$. Define $C(x) := \frac{1}{2}\log_2(1+x)$. Further define or note that

- The *nearest neighbor lattice quantizer* of Λ as

$$Q(\mathbf{x}) = \arg\min_{\lambda\in\Lambda} ||\mathbf{x}-\lambda||;$$

- The $\bmod \Lambda$ operation as $\mathbf{x} \bmod \Lambda := \mathbf{x} - Q(\mathbf{x})$;
- The *Voronoi region of* Λ as the points closer to the origin than to any other lattice point

$$\mathcal{V} := \{\mathbf{x} : Q(\mathbf{x}) = \mathbf{0}\}, \text{ sometimes denoted as } \mathcal{V}(\Lambda)$$

which is of volume $V := \text{Vol}(\mathcal{V})$ (also sometimes denoted by $V(\Lambda)$ or V_i for lattice Λ_i);

- The *second moment per dimension of a uniform distribution over* $\mathcal{V}$ as

$$\sigma^2(\Lambda) := \frac{1}{V}\cdot\frac{1}{n}\int_{\mathcal{V}} ||\mathbf{x}||^2\, d\mathbf{x};$$

- For any $\mathbf{s} \in \mathbb{R}^n$,

$$(\alpha(\mathbf{s} \bmod \Lambda)) \bmod \Lambda = (\alpha\mathbf{s}) \bmod \Lambda, \quad \alpha \in \mathbb{Z}. \tag{1}$$

$$\beta(\mathbf{s} \bmod \Lambda) = (\beta\mathbf{s}) \bmod \beta\Lambda, \quad \beta \in \mathbb{R}. \tag{2}$$

• The definitions of Rogers good and Poltyrev good sequences of lattices are stated in [7, Section II A]; we will not need these definitions explicitly. Rather, we will use the results derived from lattices with these properties.

B. Nested lattice codes

The results depend on the existence of two nested lattices Λ and Λ_c (or more precisely a sequence of n-dimensional lattices Λ and Λ_c, but we forgo this additional notation and understand Λ and Λ_c to mean a sequence of lattices of dimension n) such that $\Lambda \subseteq \Lambda_c$ with fundamental regions $\mathcal{V}, \mathcal{V}_c$ of volumes V, V_c respectively that satisfy certain properties. Here Λ is termed the *coarse* lattice which is a sublattice of Λ_c, the *fine* lattice, and hence $V \geq V_c$. When transmitting over the AWGN channel, one may use the set $\mathcal{C}_{\Lambda_c,\mathcal{V}} = \{\Lambda_c \cap \mathcal{V}\}$ as the codebook. The coding rate R of this *nested* (Λ, Λ_c) *lattice pair* is defined as

$$R = \frac{1}{n}\log|\mathcal{C}_{\Lambda_c,\mathcal{V}}| = \frac{1}{n}\log\frac{V}{V_c}.$$

In this work, we only need one "good" nested lattice pair $\Lambda \subseteq \Lambda_c$ satisfying the same "goodness" properties as the lattices used in achieving capacity over the point-to-point AWGN channel [10] or those used in the two-way relay channel [2], [9]. Below we include a lemma stating that a nested lattice pair with the listed desired properties exists, deferring existence proofs to prior work.

Lemma 1: A sequence of n-dimensional lattices $\Lambda \subseteq \Lambda_c$ exists that satisfies the following:

1) Λ is Poltyrev good and Rogers good, while Λ_c is Poltyrev good;
2) For any $\delta > 0$, $1-\delta \leq \sigma^2(\Lambda) \leq 1$ for sufficiently large n;
3) The coding rate R may approach any value as $n \to \infty$;
4) There exists a one-to-one mapping between $\mathbb{F}_{P_{prime}}$ and $\mathcal{C}_{\Lambda_c,\mathcal{V}}$ denoted by $\phi(\cdot)$ given in (3).

Proof: Points 1, 2 and 3 follow immediately from example [2, Theorem 2], which in turn is based on [10], [25]. We elaborate on point 4 as we will need $\phi(\cdot)$ in the following. Suppose the coarse lattice $\Lambda = \mathbf{B}Z^n$ is both Rogers good and Poltyrev good with asymptotic second moment $\sigma^2(\Lambda) = 1$. With respect to this coarse lattice Λ, the fine lattice Λ_c is generated by Construction A [8], [10], which maps a codebook of a linear block code over a finite field into real lattice points. The generation procedure is as follows:

- Consider the vector $\mathbf{G} \in \mathbb{F}^n_{P_{prime}}$ with every element drawn from an i.i.d. uniform distribution from the finite field of order P_{prime} (a prime number) which we take to be $\mathbb{F}_{P_{prime}} = \{0, 1, 2, \ldots, P_{prime}-1\}$ under addition and multiplication modulo P_{prime}.
- The codebook $\bar{\mathcal{C}}$ of the linear block code induced by $\mathbf{G}$ is $\bar{\mathcal{C}} = \{\bar{c} = \mathbf{G}w : w \in \mathbb{F}_{P_{prime}}\}$.
- Embed this codebook into the unit cube by scaling down by a factor of P_{prime} and then place a copy at every integer vector: $\bar{\Lambda}_c = P^{-1}_{prime}\bar{\mathcal{C}} + \mathbb{Z}^n$.
- Rotate $\bar{\Lambda}_c$ by the generator matrix of the coarse lattice to obtain the desired fine lattice: $\Lambda_c = \mathbf{B}\bar{\Lambda}_c$.

Now let $\phi(\cdot)$ denote the one-to-one mapping between one element in the one dimensional finite field $w \in \mathbb{F}_{P_{prime}}$ to a point in n-dimension real space $\mathbf{t} \in \mathcal{C}_{\Lambda_c,\mathcal{V}}$:

$$\mathbf{t} = \phi(w) = (\mathbf{B}P^{-1}_{prime}\mathbf{G}w) \bmod \Lambda, \quad (3)$$

with inverse mapping $w = \phi^{-1}(\mathbf{t}) = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T(P_{prime}(\mathbf{B}^{-1}\mathbf{t} \bmod \mathbb{Z}^{\mathbf{n}}))$ (see [8, Lemma 5 and 6]). The mapping operation $\phi(\cdot)$ defined here is used in the lemmas in the next subsection. ∎

C. Technical lemmas

We first state several lemmas needed in the proposed two-way lattice-based scheme. In the following, let $\Lambda \subseteq \Lambda_c$ be a nested lattice pair satisfying the conditions in Lemma 1, and let $\mathbf{t}_{ai}$ and $\mathbf{t}_{bi} \in \mathcal{C}_{\Lambda_c,\mathcal{V}}$ be generated from w_{ai} and $w_{bi} \in \mathbb{F}_{P_{prime}}$ as $\mathbf{t}_{ai} = \phi(w_{ai}), \mathbf{t}_{bi} = \phi(w_{bi})$. Furthermore, let $\alpha, \alpha_i, \beta_i \in \mathbb{Z}^+$ such that $\frac{\alpha}{P_{prime}}, \frac{\alpha_i}{P_{prime}}, \frac{\beta_i}{P_{prime}} \notin \mathbb{Z}$ and $\theta \in \mathbb{R}^+$. We use $\oplus$, $\otimes$ and $\ominus$ to denote modulo P_{prime} addition, multiplication, and subtraction over the finite field $\mathbb{F}_{P_{prime}}$, which is isomorphic to $\mathbb{Z}_{P_{prime}}$.

Lemma 2: There exists an one-to-one mapping between $\mathbf{v} = (\sum_i \alpha_i\theta\mathbf{t}_{ai} + \sum_i \beta_i\theta\mathbf{t}_{bi}) \bmod \theta\Lambda$ and $\mathbf{u} = \bigoplus_i \alpha_i w_{ai} \oplus \bigoplus_i \beta_i w_{bi}$.[4]

Proof: The proof follows from [8, Lemma 6], where it is shown that there is an one-to-one mapping between $\theta^{-1}\mathbf{v} = (\sum_i \alpha_i\mathbf{t}_{ai} + \sum_i \beta_i\mathbf{t}_{bi}) \bmod \Lambda$ and $\mathbf{u}' = \bigoplus_i \alpha'_i w_{ai} \oplus \bigoplus_i \beta'_i w_{bi}$, where $\alpha'_i = \alpha_i \bmod P_{prime}$ and $\beta'_i = \beta_i \bmod P_{prime}$. This one-to-one mapping is $\theta^{-1}\mathbf{v} = \phi^{-1}(\mathbf{u}')$ and $\mathbf{u}' = \phi(\theta^{-1}\mathbf{v})$. Observe that $\bigoplus_i \alpha_i w_{ai} \oplus \bigoplus_i \beta_i w_{bi} = \bigoplus_i \alpha'_i w_{ai} \oplus \bigoplus_i \beta'_i w_{bi}$ by the properties of modulo addition and multiplication. Thus, the one-to-one mapping between $\mathbf{v}$ and $\mathbf{u}$ is $\mathbf{v} = \theta\phi^{-1}(\mathbf{u})$ and $\mathbf{u} = \phi(\theta^{-1}\mathbf{v})$. ∎

The following lemma ensures that an unique $\alpha \otimes w$ can be determined given w and less trivially, that w may be determined given $\alpha \otimes w$, which is the point of the proof. This will be needed to decode messages from scaled combinations of messages in our two-way relaying scheme.

Lemma 3: There exists an one-to-one mapping between $\alpha \otimes w$ and w.

Proof: Recall that $\frac{\alpha}{P_{prime}} \notin \mathbb{Z}$. Suppose $\alpha \otimes w_1 = \alpha \otimes w_2$. Then $\alpha(w_1 - w_2) = \kappa P_{prime}$ for some integer κ. Re-writing this, we have $\frac{\alpha}{P_{prime}}(w_1 - w_2) = \kappa$ for some integer κ. This implies that either $\frac{\alpha}{P_{prime}} \in \mathbb{Z}$ or $\frac{w_1 - w_2}{P_{prime}} \in \mathbb{Z}$. Since $\frac{\alpha}{P_{prime}} \notin \mathbb{Z}$, $\frac{w_1-w_2}{P_{prime}} \in \mathbb{Z}$. Thus $w_1 - w_2 = 0$ since $-(P_{prime}-1) \leq w_1 - w_2 \leq P_{prime} - 1$. Hence $w_1 = w_2$. ∎

Lemma 4: If w_{ai} and w_{bi} are uniformly distributed over $\mathbb{F}_{P_{prime}}$, then $(\sum_i \alpha_i\theta\mathbf{t}_{ai} + \sum_i \beta_i\theta\mathbf{t}_{bi}) \bmod \theta\Lambda$ is uniformly distributed over $\{\theta\Lambda_c \cap \mathcal{V}(\theta\Lambda)\}$.

Proof: As in the proof of Lemma 2, $(\sum_i \alpha_i\theta\mathbf{t}_{ai} + \sum_i \beta_i\theta\mathbf{t}_{bi}) \bmod \theta\Lambda = \theta((\sum_i \alpha_i\mathbf{t}_{ai} + \sum_i \beta_i\mathbf{t}_{bi}) \bmod \Lambda) = \theta\phi(\bigoplus_i \alpha_i w_{ai} \oplus \bigoplus_i \beta_i w_{bi})$. Since w_{ai} and w_{bi} are uniformly distributed over $\mathbb{F}_{P_{prime}}$, $\alpha_i w_{ai}$ and $\beta_i w_{bi}$ are uniformly distributed over $\mathbb{F}_{P_{prime}}$ by Lemma 3. Then,

[4] Notice we have abused notation slightly since $\alpha_i, \beta_i \in \mathbb{Z}$ while $w_{ai}, w_{bi} \in \mathbb{F}_{P_{prime}}$, but that we understand summations where both appear to be over $\mathbb{Z}_{P_{prime}}$, which is isomorphic to $\mathbb{F}_{P_{prime}}$.

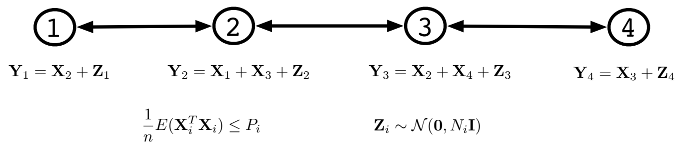

Fig. 1. The Gaussian two-way two-relay line network channel model.

$\bigoplus_i \alpha_i w_{ai} \oplus \bigoplus_i \beta_i w_{bi}$ is uniformly distributed over $\mathbb{F}_{P_{prime}}$, and $\phi(\bigoplus_i \alpha_i w_{ai} \oplus \bigoplus_i \beta_i w_{bi})$ is uniformly distributed over $\{\Lambda_c \cap \mathcal{V}(\Lambda)\}$, and finally $\theta\phi(\bigoplus_i \alpha_i w_{ai} \oplus \bigoplus_i \beta_i w_{bi})$ is uniformly distributed over $\{\theta\Lambda_c \cap \mathcal{V}(\theta\Lambda)\}$. ■

Lemma 5: Let $\mathbf{X}_a = \alpha\theta\mathbf{t}_a = \alpha\theta\phi(w_a) \in \{\alpha\theta\Lambda_c \cap \mathcal{V}(\alpha\theta\Lambda)\}$ and $\mathbf{X}_b = \theta\mathbf{t}_b = \theta\phi(w_b) \in \{\theta\Lambda_c \cap \mathcal{V}(\theta\Lambda)\}$ where $w_a, w_b \in \mathbb{F}_{P_{prime}}$. From the received signal $\mathbf{Y} = \mathbf{X}_a + \mathbf{X}_b + \mathbf{Z}$ where $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \sigma_z^2\mathbf{I})$ one may decode $(\alpha\theta\mathbf{t}_a + \theta\mathbf{t}_b) \bmod \alpha\theta\Lambda$ with arbitrary low probability of error as $n \to \infty$ at rates

$$R < \left[\frac{1}{2}\log\frac{\sigma^2(\alpha\theta\Lambda)}{\sigma_z^2}\right]^+ \quad (4)$$

$$R < \left[\frac{1}{2}\log\frac{\sigma^2(\theta\Lambda)}{\sigma_z^2}\right]^+. \quad (5)$$

Proof: The proof follows [2, Theorem 3] or [22, Theorem 3] which depends heavily of [10, Lemma 6, 11], except that for simplicity, we do not use dithers or MMSE scaling (i.e. we take $\alpha = 1$ in the notation of [22, Theorem 3]). Rather, the receiver processes the received signal as

$$\begin{aligned}\mathbf{Y} \bmod \alpha\theta\Lambda &= \mathbf{X}_a + \mathbf{X}_b + \mathbf{Z} \bmod \alpha\theta\Lambda \\ &= \alpha\theta\mathbf{t}_a + \theta\mathbf{t}_b + \mathbf{Z} \bmod \alpha\theta\Lambda.\end{aligned}$$

To decode $(\alpha\theta\mathbf{t}_a + \theta\mathbf{t}_b) \bmod \alpha\theta\Lambda$, the effective noise is given by $\mathbf{Z}$ with variance σ_z^2 rather than the equivalent noise after MMSE scaling as in [2, pg. 5491]. All other steps remain identical. The effective signal-to-noise ratios are $SNR_a = \frac{\sigma^2(\alpha\theta\Lambda_a)}{\sigma_z^2}$ and $SNR_b = \frac{\sigma^2(\theta\Lambda_b)}{\sigma_z^2}$, resulting in (4), (5). ■

Lemma 6: Let $\mathbf{X} = \theta\mathbf{t} = \theta\phi(w) \in \{\theta\Lambda_c \cap \mathcal{V}(\theta\Lambda)\}$ From the received signal $\mathbf{Y} = \mathbf{X} + \mathbf{Z}$ where $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \sigma_z^2\mathbf{I})$, one may decode $\theta\mathbf{t}$ with arbitrary low probability of error as $n \to \infty$ at rate

$$R < \frac{1}{2}\log\frac{\sigma^2(\theta\Lambda)}{\sigma_z^2}. \quad (6)$$

Proof: The proof generally follows [10, Theorem 5] and the encoding/decoding in [10, Section IV A], but without the dithers or MMSE scaling. Rather, the receiver processes the received signal as

$$\begin{aligned}\mathbf{Y} \bmod \Lambda &= \mathbf{X} + \mathbf{Z} \bmod \Lambda \\ &= \mathbf{t} + \mathbf{Z} \bmod \Lambda.\end{aligned}$$

To decode $\mathbf{t}$, the effective noise is $\mathbf{Z}$ with variance σ_z^2 rather than an equivalent noise after MMSE as in [10, Theorem 5]. The effective signal-to-noise ratio is thus $SNR = \frac{\sigma^2(\theta\Lambda)}{\sigma_z^2}$, and we obtain (6). ■

III. Channel Model

The Gaussian two-way two-relay line network describes a wireless communication scenario where two full-duplex source nodes (Node 1 and 4) simultaneously communicate with each other through two full-duplex relays (Node 2 and 3) and multiple hops as shown in Figure 1. Extensions to multiple relays are straightforward. Each node can only communicate with its neighboring nodes. The channel model may be expressed as (all bold symbols are n dimensional)

$$\begin{aligned}\mathbf{Y}_1 &= \mathbf{X}_2 + \mathbf{Z}_1 \\ \mathbf{Y}_2 &= \mathbf{X}_1 + \mathbf{X}_3 + \mathbf{Z}_2 \\ \mathbf{Y}_3 &= \mathbf{X}_2 + \mathbf{X}_4 + \mathbf{Z}_3 \\ \mathbf{Y}_4 &= \mathbf{X}_3 + \mathbf{Z}_4\end{aligned}$$

where $\mathbf{Z_i}$ ($i \in \{1,2,3,4\}$) is an i.i.d. Gaussian noise vector with variance N_i: $\mathbf{Z}_i \sim \mathcal{N}(\mathbf{0}, N_i\mathbf{I})$, and the input $\mathbf{X}_i$ is subject to the transmit power constraint P_i: $\frac{1}{n}E(\mathbf{X}_i^T\mathbf{X}_i) \leq P_i$. Note that we are assuming perfect self-interference cancellation at the full duplex nodes. That is, we assume that we can always perfectly subtract off the signal transmitted by the node itself from its received signal, and these self-interference terms are hence omitted from the channel model expression. Also note the arbitrary power constraints and noise variances but unit channel gains, making this model without loss of generality.

An $(2^{nR_a}, 2^{nR_b}, n)$ code for the Gaussian two-way two-relay channel consists of the two sets of messages w_a, w_b uniformly distributed over $\mathcal{M}_a := \{1, 2, \cdots, 2^{nR_a}\}$ and $\mathcal{M}_b := \{1, 2, \cdots, 2^{nR_b}\}$ respectively, and two encoding functions $X_1^n : \mathcal{M}_a \to \mathbb{R}^n$ (shortened to $\mathbf{X_1}$) and $X_4^n : \mathcal{M}_b \to \mathbb{R}^n$ (shortened to $\mathbf{X_4}$), satisfying the power constraints P_1 and P_4 respectively, two sets of relay functions $\{f_{k,j}\}_{j=1}^n$ ($k = 2, 3$) such that the relay channel input at time j is a function of the previously received relay channel outputs from channel uses 1 to $j-1$, $X_{k,j} = f_{k,j}(Y_{k,1}, \cdots, Y_{k,j-1})$, and finally two decoding functions $g_1 : \mathcal{Y}_1^n \times \mathcal{M}_a \to \mathcal{M}_b$ and $g_4 : \mathcal{Y}_4^n \times \mathcal{M}_b \to \mathcal{M}_a$ which yield the message estimates $\hat{w}_b := g_1(Y_1^n, w_a)$ and $\hat{w}_a := g_2(Y_4^n, w_b)$ respectively. We define the average probability of error of the code to be $P_{n,e} := \frac{1}{2^{n(R_a+R_b)}}\sum_{w_a \in \mathcal{M}_a, w_b \in \mathcal{M}_b} \Pr\{(\hat{w_a}, \hat{w_b}) \neq (w_a, w_b) | (w_a, w_b) \text{ sent}\}$. The rate pair (R_a, R_b) is then said to be achievable by the two-way two-relay channel if, for any $\epsilon > 0$ and for sufficiently large n, there exists an $(2^{nR_a}, 2^{nR_b}, n)$ code such that $P_{n,e} < \epsilon$. The capacity region of the Gaussian two-way two-relay channel is the supremum of the set of achievable rate pairs.

IV. Lattice Codes in the BC Phase of the two-way relay channel

The work [2], [9] introduces a two-phase lattice scheme for the Gaussian two-way relay channel $1 \leftrightarrow 2 \leftrightarrow 3$, where two user nodes 1 and 3 exchange information through a single relay node 2 (all definitions are analogous to the previous section): the Multiple-access Channel (MAC) phase and the Broadcast Channel (BC) phase. In the MAC phase, the relay receives a noisy version of the sum of two signals from both users as in a multiple access channel (MAC). If the codewords are from nested lattice codebooks, the relay may decode the sum of the two codewords directly without decoding them individually. This is sufficient for this channel, as then, in the BC phase, the relay may broadcast the sum of the codewords to both users who may determine the other message using knowledge of their own transmitted message. In the scheme of [2], the relay re-encodes the decoded sum of the codewords into a codeword from an i.i.d. random codebook while [9] uses a lattice codebook in the downlink.

In extending the schemes of [2], [9] to multiple relays we would want to use lattice codebooks in the BC phase, as in [9]. This would, for example, allow the signal sent by Node 2 to be aligned with Node 4's transmitted signal (aligned is used to mean that the two codebooks are nested) in the two-way two-relay line network $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$ and hence enable the decoding of the sum of codewords again at Node 3. However, the scheme of [9] is only applicable to channels in which the SNR from the users to the relay are symmetric, i.e. $\frac{P_1}{N_2} = \frac{P_3}{N_2}$. In this case the relay can simply broadcast the decoded (and possibly scaled) sum of codewords sum without re-encoding it. Thus, before tackling the two-way two-relay channel, we first devise a lattice-coding scheme for the BC phase in the two-way relay channel with *arbitrary* uplink SNRs $\frac{P_1}{N_2} \neq \frac{P_3}{N_2}$.

A. Lattice codes for the BC phase for the general Gaussian two-way relay channel

In this section, we design a lattice coding scheme for the BC phase (lattices are also used in the MAC phase) of the Gaussian two-way (single-relay) channel with arbitrary uplink SNR – i.e. not restricted to symmetric SNRs as in [9]. For simplicity, to demonstrate the central idea of a lattice-based BC phase which is going to be used in the two-way two-relay line network in Section V we do not use dithers nor MMSE scaling as in [2], [9], [10][5].

The channel model is the same as in [2]: two users Node 1 and 3 communicate with each other through the relay Node 2. The channel model is expressed as

$$\mathbf{Y}_1 = \mathbf{X}_2 + \mathbf{Z}_1$$
$$\mathbf{Y}_2 = \mathbf{X}_1 + \mathbf{X}_3 + \mathbf{Z}_2$$
$$\mathbf{Y}_3 = \mathbf{X}_2 + \mathbf{Z}_3$$

where $\mathbf{Z_i}$ ($i \in \{1, 2, 3\}$) is an i.i.d. Gaussian noise vector with variance N_i: $\mathbf{Z}_i \sim \mathcal{N}(\mathbf{0}, N_i\mathbf{I})$, and the input $\mathbf{X}_i$ is subject to the transmitting power constraint P_i: $\frac{1}{n}E(\mathbf{X}_i^T\mathbf{X}_i) \leq P_i$. Similar definitions of codes and achievability as in Section III are assumed.

We assume that $P_1 = N^2p^2$ and $P_3 = p^2$ where $p \in \mathbb{R}$ and $N \in \mathbb{Z}$. This assumption will be generalized to arbitrary power constraints in the next section. We focus on the symmetric rate for the two-way two-relay channel, i.e. when the coding rates of the two messages are identical.

[5]Dithers and MMSE scaling allows one to go from achieving rates proportional to $\log(SNR)$ to $\log(1 + SNR)$ in decoding a single lattice codeword and to $\log(1/2+SNR)$ in decoding the sum of lattice codewords. However, we initially forgo the "1+" or "1/2+" term for simplicity and as not to clutter the main idea with additional dithers and MMSE scaling.

Codebook generation: Consider the messages $w_a, w_b \in \mathbb{F}_{P_{prime}} = \{0, 1, 2, \ldots, P_{prime} - 1\}$. P_{prime} is a large prime number such that $P_{prime} = [2^{nR_{sym}}]$, where R_{sym} is the symmetric coding rate and $[\,]$ denotes rounding to the nearest prime ($P_{prime} = [2^{nR_{sym}}] \to \infty$ as $n \to \infty$ since there are infinitely many primes). The two users Node 1 and 2 send the codewords $\mathbf{X}_1 = Np\mathbf{t}_a = Np\phi(w_a)$ and $\mathbf{X}_2 = p\mathbf{t}_b = p\phi(w_b)$ where $\phi(\cdot)$ is defined in (3) in Section II-B with the nested lattices $\Lambda \subseteq \Lambda_c$ satisfying the conditions of Lemma 1. Notice that the users' codebooks are scaled versions of the codebook $\mathcal{C}_{\Lambda_c,\mathcal{V}}$. The symmetric coding rate is then $R_{sym} := \frac{1}{n}\log\frac{V(\Lambda)}{V(\Lambda_c)}$.

In the MAC phase, the relay receives $\mathbf{Y}_2 = \mathbf{X}_1 + \mathbf{X}_3 + \mathbf{Z}_2$ and decodes $(Np\mathbf{t}_a + p\mathbf{t}_b) \bmod Np\Lambda$ with arbitrarily low probability of error as $n \to \infty$ with rate constraints

$$R_{sym} < \left[\frac{1}{2}\log\left(\frac{P_1}{N_2}\right)\right]^+$$
$$R_{sym} < \left[\frac{1}{2}\log\left(\frac{P_3}{N_2}\right)\right]^+$$

according to Lemma 5.

In the BC phase, if, mimicking the steps of [9], the relay simply broadcasts the scaled version of $(Np\mathbf{t}_a + p\mathbf{t}_b) \bmod Np\Lambda$

$$\frac{\sqrt{P_2}}{Np}((Np\mathbf{t}_a + p\mathbf{t}_b) \bmod Np\Lambda) = \left(\sqrt{P_2}\mathbf{t}_a + \frac{\sqrt{P_2}}{N}\mathbf{t}_b\right) \bmod \sqrt{P_2}\Lambda,$$

we would achieve the rate $R_{sym} < [\frac{1}{2}\log\frac{P_2}{N_3}]^+$ for the direction $2 \to 3$ and the rate $R_{sym} < [\frac{1}{2}\log\frac{P_2}{NN_1}]^+$ for the $1 \leftarrow 2$ direction. While the rate constraint for the direction $2 \to 3$ is as large as expected, the rate constraint for the direction $1 \leftarrow 2$ does not fully utilize the power at the relay, i.e. the codeword $\mathbf{t}_b$ appears to use only the power P_2/N rather than the full power P_2. One would thus want to somehow transform the decoded sum $(Np\mathbf{t}_a + p\mathbf{t}_b) \bmod Np\Lambda$ such that both $\mathbf{t}_a$ and $\mathbf{t}_b$ of the transformed signal would be uniformly distributed over $\mathcal{V}(\sqrt{P_2}\Lambda)$. Notice that the relay can only operate on $(Np\mathbf{t}_a + p\mathbf{t}_b) \bmod Np\Lambda$ rather than $Np\mathbf{t}_a$ and $p\mathbf{t}_b$ individually.

Folding and Scaling: To alleviate this problem, we may fold the decoded signal with the operation $\bmod\ p\Lambda$ and then scale the result to meet the desired transmit power constraint. Intuitively, although decoding is with respect to the coarse lattice, we fold it back to the fine lattice to fully utilize the relay's transmit power for broadcasting the codewords in both directions. Essentially, we reduce the cardinality of the transmitted codewords or compress the decoded signal at the relay but keep the information lossless for both direction when side information is accounted for.

The procedure is as follows:

1) Take $\bmod\ p\Lambda$ on the decoded signal to obtain

$$(Np\mathbf{t}_a + p\mathbf{t}_b) \bmod Np\Lambda \bmod p\Lambda = (Np\mathbf{t}_a + p\mathbf{t}_b) \bmod p\Lambda$$

according to the operation rule in (1).

2) Re-scale the signal to be of second moment P_2 as

$$\frac{\sqrt{P_2}}{p}((Np\mathbf{t}_a + p\mathbf{t}_b) \bmod p\Lambda) = (N\sqrt{P_2}\mathbf{t}_a + \sqrt{P_2}\mathbf{t}_b) \bmod \sqrt{P_2}\Lambda$$

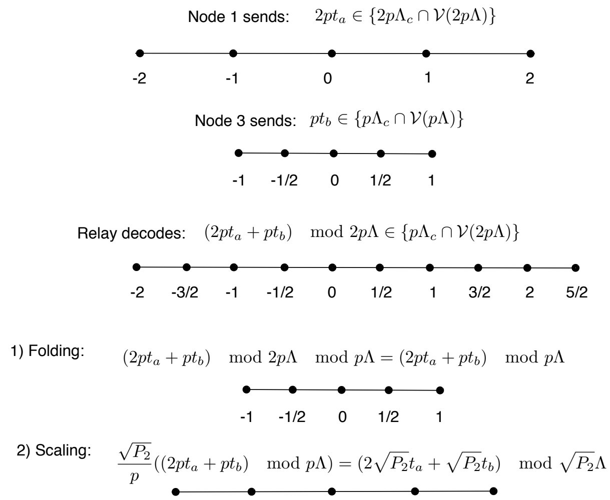

Fig. 2. Folding and scaling for a one-dimensional lattice.

according to the operation rule in (2). Notice that $(N\sqrt{P_2}\mathbf{t}_a + \sqrt{P_2}\mathbf{t}_b) \bmod \sqrt{P_2}\Lambda$ is uniformly distributed over $\{\sqrt{P_2}\Lambda_c \cap \mathcal{V}(\sqrt{P_2}\Lambda)\}$ by Lemma 4.

The relay broadcasts $\mathbf{X}_2 = (N\sqrt{P_2}\mathbf{t}_a + \sqrt{P_2}\mathbf{t}_b) \bmod \sqrt{P_2}\Lambda$. Notice that $(N\sqrt{P_2}\mathbf{t}_a + \sqrt{P_2}\mathbf{t}_b) \bmod \sqrt{P_2}\Lambda$ is uniformly distributed over $\{\sqrt{P_2}\Lambda_c \cap \mathcal{V}(\sqrt{P_2}\Lambda)\}$, and so its coding rate is R_{sym}. Node 1 and Node 3 receive $\mathbf{Y}_1 = \mathbf{X}_2 + \mathbf{Z}_1$ and $\mathbf{Y}_3 = \mathbf{X}_2 + \mathbf{Z}_3$ respectively and, according to Lemma 6, may decode $(N\sqrt{P_2}\mathbf{t}_a + \sqrt{P_2}\mathbf{t}_b) \bmod \sqrt{P_2}\Lambda$ at rate

$$R_{sym} < \left[\frac{1}{2}\log\frac{P_2}{N_1}\right]^+$$
$$R_{sym} < \left[\frac{1}{2}\log\frac{P_2}{N_3}\right]^+.$$

Nodes 1 and 3 then map the decoded $(NP_2\mathbf{t}_a + P_2\mathbf{t}_b) \bmod P_2\Lambda$ to $Nw_a \oplus w_b$ by Lemma 2. With side information w_a, Node 1 may then determine w_b; likewise with side information w_b, Node 3 can obtain Nw_a and then determine w_a by Lemma 3.

Example of 1-D Folding and Scaling. One of the key steps in the above proof is the folding and scaling operation. To obtain some intuition and insight, the folding and scaling procedures are illustrated in Figure 2 for a simple one-dimensional lattice (though we operate in n-dimensions in our achievability proof). In this simple example, Node 1 sends $2pt_a \in \{2p\Lambda_c \cap \mathcal{V}(2p\Lambda)\} = \{-2, -1, 0, 1, 2\}$ and Node 3 sends $pt_b \in \{p\Lambda_c \cap \mathcal{V}(p\Lambda)\} = \{-1, -1/2, 0, 1/2, 1\}$. The relay decodes

$$\begin{aligned}(2pt_a + pt_b) \bmod 2p\Lambda &\in \{p\Lambda_c \cap \mathcal{V}(2p\Lambda)\}\\ &= \{-2, -3/2, -1, -1/2, 0, 1/2, 1, 3/2, 2, 5/2\}\end{aligned}$$

and folds it into $(2pt_a + pt_b) \bmod 2p\Lambda \bmod p\Lambda = (2pt_a + pt_b) \bmod p\Lambda \in \{p\Lambda_c \cap \mathcal{V}(p\Lambda)\} = \{-1, -1/2, 0, 1/2, 1\}$. Notice $|\{p\Lambda_c \cap \mathcal{V}(p\Lambda)\}| < |\{p\Lambda_c \cap \mathcal{V}(2p\Lambda)\}|$, and the transformed signal is thus easier (larger minimum distance, fewer points) to forward.

V. Two-way two-relay line network

We now consider the full-duplex two-way two-relay line network. For this channel model we first obtain an achievable rate region, valid when the relay powers satisfy either 1) $P_1 = p^2$, $P_2 = M^2q^2$, $P_3 = N^2p^2$ and $P_4 = q^2$ (Theorem 7), or 2) three permutations thereof as described in Lemma 8, where $p, q \in \mathbb{R}^+$ and $M, N \in \mathbb{Z}^+$. We use this to obtain an achievable rate region for the general (arbitrary powers) Gaussian two-way two-relay line network which we show is to within $\frac{1}{2}\log(3)$ bits/s/Hz per user of the symmetric capacity in Theorem 9.

Theorem 7: For the channel model described in Section III, if $P_1 = p^2$, $P_2 = M^2q^2$, $P_3 = N^2p^2$ and $P_4 = q^2$, where $p, q \in \mathbb{R}^+$ and $M, N \in \mathbb{Z}^+$ the following rate region

$$\begin{aligned}R_a, R_b < \min\Bigg(&\left[\frac{1}{2}\log\left(\frac{P_1}{N_2}\right)\right]^+, \left[\frac{1}{2}\log\left(\frac{P_2}{N_3}\right)\right]^+, \left[\frac{1}{2}\log\left(\frac{P_3}{N_4}\right)\right]^+,\\ &\left[\frac{1}{2}\log\left(\frac{P_4}{N_3}\right)\right]^+, \left[\frac{1}{2}\log\left(\frac{P_3}{N_2}\right)\right]^+, \left[\frac{1}{2}\log\left(\frac{P_2}{N_1}\right)\right]^+\Bigg)\end{aligned} \tag{7}$$

is achievable using lattice codes.

Remark 1: Notice there are some redundant terms in the expression (7) as we assume $P_3 \geq P_1$ and $P_2 \geq P_4$. But we keep these terms for convenience and coherence with the following theorems.

Proof:

Codebook generation: We consider a nested lattice pair $\Lambda \subseteq \Lambda_c$ with corresponding codebook $\mathcal{C}_{\Lambda_c,\mathcal{V}} = \{\Lambda_c \cap \mathcal{V}(\Lambda)\}$ satisfying Lemma 1, and two messages $w_a, w_b \in \mathbb{F}_{P_{prime}} = \{0, 1, 2, \ldots, P_{prime} - 1\}$ in which P_{prime} is a large prime number such that $P_{prime} = [2^{nR_{sym}}]$ (R_{sym} is the coding rate). The codewords associated with the messages w_a and w_b are $\mathbf{t}_a = \phi(w_a)$ and $\mathbf{t}_b = \phi(w_b)$, where the mapping $\phi(\cdot)$ from $\mathbb{F}_{P_{prime}}$ to $\mathcal{C}_{\Lambda_c,\mathcal{V}} \in \mathbb{R}^n$ is defined in (3) in Section II-B.

Encoding and decoding steps: We use a block Markov Encoding/Decoding scheme where Node 1 and 4 transmit a new message w_{ai} and w_{bi}, respectively, at the beginning of block i. To satisfy the transmit power constraints, Node 1 and 4 send the scaled codewords $\mathbf{X}_{1i} = p\mathbf{t}_{ai} = p\phi(w_{ai}) \in \{p\Lambda_c \cap \mathcal{V}(p\Lambda)\}$ and $\mathbf{X}_{4i} = q\mathbf{t}_{bi} = q\phi(w_{bi}) \in \{q\Lambda_c \cap \mathcal{V}(q\Lambda)\}$ respectively in block i. Node 2 and 3 send $\mathbf{X}_{2i}$ and $\mathbf{X}_{3i}$, and Node j ($j = \{1, 2, 3, 4\}$) receives $\mathbf{Y}_{ji}$ in block i. The procedure of the first few blocks (the initialization steps) are described and then a generalization is made. We note that in general the coding rates R_a for w_a and R_b for w_b may be different, as long as $R_{sym} = \max(R_a, R_b)$, since we may send dummy messages to make the two coding rates equal.

Block 1: Node 1 and 4 send new codewords $\mathbf{X}_{11} = p\mathbf{t}_{a1}$ and $\mathbf{X}_{41} = q\mathbf{t}_{b1}$ to Node 2 and 3 respectively. Node 2 and 3 can decode the transmitted codeword with vanishing probability of error if

$$R_{sym} < \left[\frac{1}{2}\log\left(\frac{P_1}{N_2}\right)\right]^+ \tag{8}$$

$$R_{sym} < \left[\frac{1}{2}\log\left(\frac{P_4}{N_3}\right)\right]^+ \tag{9}$$

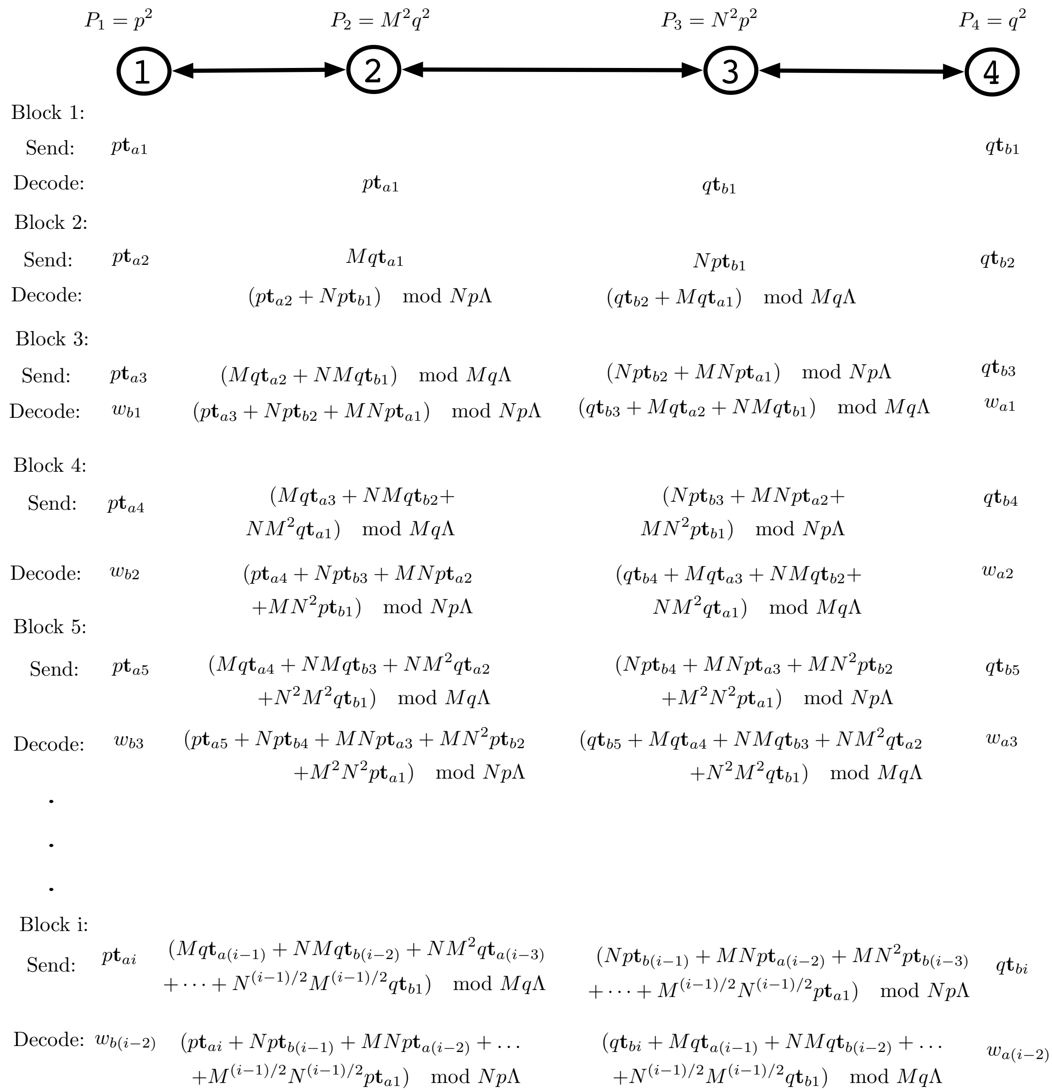

Fig. 3. Multi-phase block Markov achievability strategy for Theorem 7.

according to Lemma 6.

Block 2: Node 1 and 4 send their respective new codewords $\mathbf{X}_{12} = p\mathbf{t}_{a2}$ and $\mathbf{X}_{42} = q\mathbf{t}_{b2}$, while Node 2 and 3 broadcast $\mathbf{X}_{22} = Mq\mathbf{t}_{a1}$ and $\mathbf{X}_{32} = Np\mathbf{t}_{b1}$ received in the last block. Note they are scaled to fully utilize the transmit power $P_2 = M^2q^2$ and $P_3 = N^2p^2$. Node 2 receives $\mathbf{Y}_{22} = \mathbf{X}_{12} + \mathbf{X}_{32} + \mathbf{Z}_{22}$ and decodes $(p\mathbf{t}_{a2} + Np\mathbf{t}_{b1}) \mod Np\Lambda$ with arbitrarily low probability of error if R satisfies (8) and

$$R_{sym} < \left[\frac{1}{2}\log\left(\frac{P_3}{N_2}\right)\right]^+. \tag{10}$$

Similarly, Node 3 can decode $(q\mathbf{t}_{b2} + Mq\mathbf{t}_{a1}) \mod Mq\Lambda$ subject to (9) and

$$R_{sym} < \left[\frac{1}{2}\log\left(\frac{P_2}{N_3}\right)\right]^+. \tag{11}$$

Block 3:

- *Encoding:* Node 1 and 4 send new codewords as in the previous blocks. Node 2 further processes its decoded codewords combination by first folding the signal:

$$(p\mathbf{t}_{a2} + Np\mathbf{t}_{b1}) \mod Np\Lambda \mod p\Lambda = (p\mathbf{t}_{a2} + Np\mathbf{t}_{b1}) \mod p\Lambda$$

and then scaling it to fully utilize the transmit power $P_2 = M^2q^2$ as

$$\frac{Mq}{p}(p\mathbf{t}_{a2}+Np\mathbf{t}_{b1}) \mod p\Lambda = (Mq\mathbf{t}_{a2}+NMq\mathbf{t}_{b1}) \mod Mq\Lambda.$$

It then broadcasts $\mathbf{X}_{23} = (Mq\mathbf{t}_{a2} + NMq\mathbf{t}_{b1}) \mod Mq\Lambda$. Notice that since

$$(Mq\mathbf{t}_{a2} + NMq\mathbf{t}_{b1}) \mod Mq\Lambda \in \{Mq\Lambda_c \cap \mathcal{V}(Mq\Lambda)\}$$

according to Lemma 4, its coding rate is R_{sym}. Similarly, Node 3 broadcasts $\mathbf{X}_{33} = (Np\mathbf{t}_{b2} + MNp\mathbf{t}_{a1}) \mod Np\Lambda$ again at coding rate R_{sym}.

- *Decoding:* At the end of this block, Node 2 is able to decode $(p\mathbf{t}_{a3} + Np\mathbf{t}_{b2} + MNp\mathbf{t}_{a1}) \mod Np\Lambda$ with rate constraints (8) and (10) according to Lemma 5, and Node 3 decodes $(q\mathbf{t}_{b3} + Mq\mathbf{t}_{a2} + NMq\mathbf{t}_{b1}) \mod Mq\Lambda$ with constraints (11) and (9). Node 1 decodes

$(Mq\mathbf{t}_{a2} + NMq\mathbf{t}_{b1}) \bmod Mq\Lambda$ sent by Node 2 as in the point-to-point channel with rate constraint

$$R_{sym} < \left[\frac{1}{2}\log\left(\frac{P_2}{N_1}\right)\right]^+ \qquad (12)$$

according to Lemma 6. From the decoded $(Mq\mathbf{t}_{a2} + NMq\mathbf{t}_{b1}) \bmod Mq\Lambda$, it obtains $w_{a2} \oplus N w_{b1}$ (Lemma 2). With its own information w_{a2}, Node 1 can then obtain $N \otimes w_{b1} = w_{a2} \oplus N w_{b1} \ominus w_{a2}$, which may be mapped to w_{b1} since P_{prime} is a prime number (Lemma 3). Let $[x]$ denote the smallest prime number larger than x. Notice $P_{prime} = [2^{nR_{sym}}] \to \infty$ as $n \to \infty$, so $N \ll P_{prime} = [2^{nR}]$ and $\frac{N}{P_{prime}} \notin \mathbb{Z}$. Similarly, Node 4 can decode w_{a1} with rate constraint

$$R_{sym} < \left[\frac{1}{2}\log\left(\frac{P_2}{N_3}\right)\right]^+. \qquad (13)$$

Block 4 and 5 proceed in a similar manner, as shown in Figure 3.

Block i: To generalize, in block i (assume i is odd)[6],

- *Encoding:* Node 1 and 4 send new messages $\mathbf{X}_{1i} = p\mathbf{t}_{ai}$ and $\mathbf{X}_{4i} = q\mathbf{t}_{bi}$ respectively. Node 2 and 3 broadcast

$$\mathbf{X}_{2i} = (Mq\mathbf{t}_{a(i-1)} + NMq\mathbf{t}_{b(i-2)} + NM^2q\mathbf{t}_{a(i-3)} + N^2M^2q\mathbf{t}_{b(i-4)} + \cdots + N^{(i-1)/2}M^{(i-1)/2}q\mathbf{t}_{b1}) \bmod Mq\Lambda$$
$$\mathbf{X}_{3i} = (Np\mathbf{t}_{b(i-1)} + MNp\mathbf{t}_{a(i-2)} + MN^2p\mathbf{t}_{b(i-3)} + M^2N^2p\mathbf{t}_{a(i-4)} + \cdots + M^{(i-1)/2}N^{(i-1)/2}p\mathbf{t}_{a1}) \bmod Np\Lambda.$$

- *Decoding:* Node 1 decodes the codeword from Node 2 with rate constraint (12) (Lemma 6) and maps it to $w_{a(i-1)} \oplus N w_{b(i-2)} \oplus NM w_{a(i-3)} \oplus N^2M w_{b(i-4)} \oplus \cdots \oplus N^{(i-1)/2}M^{(i-1)/2-1}w_{b1}$ (Lemma 2). With its own messages w_{ai} $(\forall i)$ and the messages it decoded previously $\{w_{b1}, w_{b2}, \ldots, w_{b(i-3)}\}$, Node 1 can obtain $N \otimes w_{b(i-2)}$ and determine $w_{b(i-2)}$ accordingly (Lemma 3). Similarly, Node 4 can decode $w_{a(i-2)}$ subject to rate constraint (13).
- *Folding and Scaling:* In this block i, Node 2 also decodes

$$(p\mathbf{t}_{ai} + Np\mathbf{t}_{b(i-1)} + MNp\mathbf{t}_{a(i-2)} + MN^2p\mathbf{t}_{b(i-3)} + M^2N^2p\mathbf{t}_{a(i-4)} + \cdots + M^{(i-1)/2}N^{(i-1)/2}p\mathbf{t}_{a1}) \bmod Np\Lambda$$

from its received signal $\mathbf{Y}_{2i} = \mathbf{X}_{1i} + \mathbf{X}_{3i} + \mathbf{Z}_{2i}$ subject to rate constraints (8) and (10) (Lemma 5). It then folds the codeword combination as

$$\begin{aligned}&(p\mathbf{t}_{ai} + Np\mathbf{t}_{b(i-1)} + MNp\mathbf{t}_{a(i-2)} + \ldots \\ &\quad + M^{(i-1)/2}N^{(i-1)/2}p\mathbf{t}_{a1}) \bmod Np\Lambda \bmod p\Lambda \\ &= p\mathbf{t}_{ai} + Np\mathbf{t}_{b(i-1)} + MNp\mathbf{t}_{a(i-2)} + \ldots \\ &\quad + M^{(i-1)/2}N^{(i-1)/2}p\mathbf{t}_{a1} \bmod p\Lambda\end{aligned}$$

and scales it to fully utilize the transmit power:

$$\begin{aligned}&\frac{Mq}{p}(p\mathbf{t}_{ai} + Np\mathbf{t}_{b(i-1)} + MNp\mathbf{t}_{a(i-2)} + \ldots \\ &\quad + M^{(i-1)/2}N^{(i-1)/2}p\mathbf{t}_{a1} \bmod p\Lambda) \\ &= Mq\mathbf{t}_{ai} + NMq\mathbf{t}_{b(i-1)} + NM^2q\mathbf{t}_{a(i-2)} + \ldots \\ &\quad + N^{(i-1)/2}M^{(i-1)/2+1}q\mathbf{t}_{a1} \bmod Mq\Lambda.\end{aligned}$$

This signal will be transmitted in the next block $i+1$. Node 3 performs similar operations, decoding $q\mathbf{t}_{bi} + Mq\mathbf{t}_{a(i-1)} + NMq\mathbf{t}_{b(i-2)} + \cdots + N^{(i-1)/2}M^{(i-1)/2}q\mathbf{t}_{b1} \bmod Mq\Lambda$ subject to constraints (11) and (9), and transforms it into $Np\mathbf{t}_{bi} + MNp\mathbf{t}_{a(i-1)} + MN^2p\mathbf{t}_{b(i-2)} + \cdots + M^{(i-1)/2}N^{(i-1)/2+1}p\mathbf{t}_{b1} \bmod Mq\Lambda$, which is transmitted in the next block.

Assuming there are I blocks in total, the final achievable rate is $\frac{I-2}{I}R_{sym}$, which, as $I \to \infty$, approaches R_{sym} and we obtain (7). ■

In the above we had assumed power constraints of the form $P_1 = p^2$, $P_3 = N^2p^2$, $P_2 = M^2q^2$, and $P_4 = q^2$, where $p, q \in \mathbb{R}^+$ and $M, N \in \mathbb{Z}^+$. Analogously, we may permute some of these power constraints to achieve the same region as follows:

Lemma 8: The rates achieved in Theorem 7 may also be achieved when $P_1 = N^2p^2$, $P_3 = p^2$, $P_2 = q^2$, $P_4 = M^2q^2$, or $P_1 = N^2p^2$, $P_3 = p^2$, $P_2 = M^2q^2$, $P_4 = q^2$, or $P_1 = p^2$, $P_3 = N^2p^2$, $P_2 = q^2$, $P_4 = M^2q^2$.

Proof: The proof follows the same lines as that of Theorem 7, and consists of the steps outlined in in Figure 5 in Appendix A. In particular, since the nodes have different power constraints the scaling of the codewords is different. However, as in the previous Theorem, we only need $\mathbf{X}_1$ and $\mathbf{X}_3$ to be aligned (nested codebooks), and $\mathbf{X}_2$ and $\mathbf{X}_4$ to be aligned. As in Theorem 7, the relay nodes again decode the sum of codewords, perform the Folding and Scaling, with a new power scaling to fully utilize their transmit power, and broadcast the re-distributed sum of codewords. ■

Theorem 7 and Lemma 8 both hold for powers for which P_1/P_3 and/or P_2/P_4 are either the squares of integers or the reciprocal of the squares of integers. However, these scenarios do not cover general power constraints with arbitrary ratios. We next present an achievable rate region for arbitrary powers, obtained by appropriately clipping the power of the nodes such that the new, lower powers are indeed either the squares or the reciprocals of the squares of integers. We then show that this clipping of the power at the nodes does not result in more than a $\frac{1}{2}\log(3)$ bits/s/Hz loss in the symmetric rate.

Theorem 9: For the two-way two-relay channel with arbitrary transmit power constraints, any rates satisfying (14) for some $N, M \in \mathbb{Z}^+$ and $i \in \{1, 2, 3, 4\}$, are achievable. This rate region is within $\frac{1}{2}\log 3$ bit/Hz/s per user from the symmetric rate capacity.

Proof: Since P_1/P_3 and/or P_2/P_4 are in general neither the squares or the reciprocals of the squares of integers, we cannot directly apply Theorem 7. Instead, we first truncate the transmit powers P_i to P_i' such that P_i' satisfy either the constraints of Theorem 7 or Lemma 8. The achievable rate region then follows immediately for the reduced power constraints P_i'. For example, if $P_1 = 1$ and $P_3 = 3.6$, we may choose $P_1' = 0.9$ and $P_3' = 3.6$, so that $P_3'/P_1' = 2^2$. Optimizing over the truncated or clipped powers yields the achievable rate region stated in Theorem 9.

An outer bound to the symmetric capacity when $R_a = R_b$ of the AWGN two-way two-relay channel is given by the

[6]For i even we have analogous steps with slightly different indices as may be extrapolated from the difference between block 4 and 5 in Fig. 3, which result in the same rate constraints.

$$R_a, R_b < R_{achievable} = \max_{P_i' \le P_i, \frac{P_1'}{P_3'} = N^2 \text{ or } \frac{1}{N^2}, \frac{P_2'}{P_4'} = M^2 \text{ or } \frac{1}{M^2}} \min\left(\left[\frac{1}{2}\log\left(\frac{P_1'}{N_2}\right)\right]^+, \left[\frac{1}{2}\log\left(\frac{P_2'}{N_3}\right)\right]^+, \left[\frac{1}{2}\log\left(\frac{P_3'}{N_4}\right)\right]^+, \left[\frac{1}{2}\log\left(\frac{P_4'}{N_3}\right)\right]^+, \left[\frac{1}{2}\log\left(\frac{P_3'}{N_2}\right)\right], \left[\frac{1}{2}\log\left(\frac{P_2'}{N_1}\right)\right]^+ \right) \tag{14}$$

minimum of the all the point-to-point links, i.e.

$$R_a, R_b < R_{outer} = \min\left(C\left(\frac{P_1}{N_2}\right), C\left(\frac{P_2}{N_3}\right), C\left(\frac{P_3}{N_4}\right), C\left(\frac{P_4}{N_3}\right), C\left(\frac{P_3}{N_2}\right), C\left(\frac{P_2}{N_1}\right) \right). \tag{15}$$

To evaluate the gap between (14) and (15), let $\mathcal{A} := \{P_1', P_3' : P_1' \le P_1, P_3' \le P_3, \frac{P_1'}{P_3'} = N^2$ or $\frac{1}{N^2}\}$:

$$\begin{aligned} & R_{achievable} + \frac{1}{2}\log 3 \\ & \overset{(a)}{=} \max_{\mathcal{A}} \min\left(\left[\frac{1}{2}\log\left(\frac{P_1'}{N_2}\right)\right]^+, \left[\frac{1}{2}\log\left(\frac{P_3'}{N_4}\right)\right]^+, \left[\frac{1}{2}\log\left(\frac{P_3'}{N_2}\right)\right]^+ \right) + \frac{1}{2}\log 3 && (16) \\ & = \max_{\mathcal{A}} \max\left(\min\left(\frac{1}{2}\log\left(\frac{3P_1'}{N_2}\right), \frac{1}{2}\log\left(\frac{3P_3'}{N_4}\right), \frac{1}{2}\log\left(\frac{3P_3'}{N_2}\right) \right), \frac{1}{2}\log 3 \right) && (17) \\ & \overset{(b)}{\ge} \max\left(\min\left(\frac{1}{2}\log\left(\frac{3P_1^\star}{N_2}\right), \frac{1}{2}\log\left(\frac{3P_3^\star}{N_4}\right), \frac{1}{2}\log\left(\frac{3P_3^\star}{N_2}\right) \right), \frac{1}{2}\log 3 \right) && (18) \\ & \overset{(c)}{\ge} \min\left(\frac{1}{2}\log\left(1+\frac{P_1}{N_2}\right), \frac{1}{2}\log\left(1+\frac{P_3}{N_4}\right), \frac{1}{2}\log\left(1+\frac{P_3}{N_2}\right) \right) && (19) \\ & \ge R_{outer}. && (20) \end{aligned}$$

The first equality (a) follows from an assumption that WLOG, one of $\left[\frac{1}{2}\log\left(\frac{P_1'}{N_2}\right)\right]^+$, $\left[\frac{1}{2}\log\left(\frac{P_3'}{N_4}\right)\right]^+$, or $\left[\frac{1}{2}\log\left(\frac{P_3'}{N_2}\right)\right]^+$ is the tightest constraint. The first inequality (b) follows since the rates achieved by the optimized powers must be larger than those achieved by one particular strategy that meets the constraints – in this case the strategy that yields the $P_i^\star$ which we construct in Appendix B. Inequality (c) follows from the fact that $2P_i^\star \ge P_i$, as also shown in Appendix B. Finally, we bound $\frac{1}{2}\log\left(\frac{3P_1^\star}{N_2}\right)$ with $\frac{1}{2}\log\left(1+\frac{P_1}{N_2}\right)$ as follows: If $\frac{P_1}{N_2} \ge 2$, it follows that $\frac{P_1^\star}{N_2} \ge 1$, and hence $\frac{1}{2}\log\left(\frac{3P_1^\star}{N_2}\right) \ge \frac{1}{2}\log\left(1+\frac{P_1}{N_2}\right)$. Otherwise, $\frac{1}{2}\log\left(1+\frac{P_1}{N_2}\right) < \frac{1}{2}\log 3$. Similarly, we may bound $\frac{1}{2}\log\left(\frac{3P_3^\star}{N_4}\right)$ and $\frac{1}{2}\log\left(\frac{3P_3^\star}{N_2}\right)$ with $\frac{1}{2}\log\left(1+\frac{P_3}{N_4}\right)$ and $\frac{1}{2}\log\left(1+\frac{P_3}{N_2}\right)$ respectively. ∎

Remark 2: We achieve a constant gap to the symmetric rate capacity of $\frac{1}{2}\log(3)$ bit/s/Hz. The only other scheme we are aware of that has been shown to achieve a constant gap is noisy network coding [26] which achieves a larger gap of 1.26 bits/s/Hz to the capacity. The improvement in rates of the proposed scheme may be attributed to the removal of the noise at intermediate relays (it is a lattice-based Decode and Forward scheme) without the sum-rate constraints that would be needed in i.i.d. random coding based Decode-and-Forward schemes, though we note that the noisy network coding scheme is based on a Compress-and-Forward scheme.

VI. Extension to an arbitrary number of relays and constant gap which is independent of the number of users

We now extend our scheme to two-way line networks with an arbitrary number of full-duplex relays. In particular, we demonstrate a lattice-based achievable rate region and show that this achieves to within a constant $\frac{1}{2}\log(5)$ bits per user of the symmetric capacity, i.e. that full duplex operation at all nodes can still double the capacity (over half-duplex networks employing simple time-sharing) even in the presence of an arbitrary number of relays. We emphasize that this gap is independent of the number of relays in this line-network.

The limitation is again ensuring that the transmit powers of alternate pairs are related to each other through an integer squared relationship. For example, for the two-way Three-relay channel with five nodes: $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 5$, we may apply the same strategy by aligning the codewords between 1 and 3, 3 and 5, and 2 and 4 (i.e. nest the corresponding codebooks). To align or nest the codebooks, one may truncate the transmit powers so that $\frac{P_1'}{P_3'}$, $\frac{P_3'}{P_5'}$ and $\frac{P_2'}{P_4'}$ are either squares, or the reciprocals of squares of integers. In a general two-way line network with D nodes: $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow \ldots D-1 \leftrightarrow D$, Node 1 and D are the two users communicating with each other through the relay nodes 2 to $D-1$. Here we assume all channel gains are 1. This assumption is without loss of generality since the transmitting power constraints P_i and noise variances N_i ($i \in \{1,2,3,\ldots,D\}$) can be chosen arbitrarily.

Theorem 10: By extending our block Markov strategy, one may show that the rates in (21) are achievable for the two-way line network, where $i = \{1,2,\ldots,D\}$ and $j = \{1,2,\ldots,D-2\}$. This rate region is within $\frac{1}{2}\log 5$ bit/Hz/s per user of the symmetric rate capacity.

Proof: The proof follows almost immediately from Theorems 7 and 9. First, select the transmit power of each node as $P_i' \le P_i$ such that P_i' satisfy the conditions that all $\frac{P_j'}{P_{j+2}'}$ are integer squares or reciprocal of integer squares. By extending our block Markov strategy (omitted as no new concepts arise, it is a simple extension with much added complexity in

$$R_a, R_b < R_{achievable} = \max_{\substack{P'_i \leq P_i, \\ \frac{\sqrt{P'_j}}{\sqrt{P'_{j+2}}} \text{ or } \frac{\sqrt{P'_{j+2}}}{\sqrt{P'_j}} \in \mathbb{Z}^+}} \min_{\substack{k=\{1,2,\dots,D-1\}, \\ l=\{2,3,\dots,D\}}} \left(\left[\frac{1}{2}\log\left(\frac{P'_k}{N_{k+1}}\right)\right]^+, \left[\frac{1}{2}\log\left(\frac{P'_l}{N_{l-1}}\right)\right]^+ \right). \tag{21}$$

notation), we are able to achieve the rate region described in Theorem 10.

To show that this rate region is within $\frac{1}{2}\log 5$ bit/Hz/s per user from the symmetric rate capacity, first consider the outer bound of symmetric rate capacity, given by

$$R_{outer} = \min_{\substack{k=\{1,2,\dots,D-1\}, \\ l=\{2,3,\dots,D\}}} \left(C\left(\frac{P_k}{N_{k+1}}\right), C\left(\frac{P_l}{N_{l-1}}\right) \right). \tag{22}$$

To evaluate the gap between (21) and (22), first let $\mathcal{B} := \{P'_i : P'_i \leq P_i, \frac{\sqrt{P'_j}}{\sqrt{P'_{j+2}}} \text{ or } \frac{\sqrt{P'_{j+2}}}{\sqrt{P'_j}} \in \mathbb{Z}^+, i = \{1,3,5,\dots\}, j = \{1,3,5,\dots,D-2\}\}$ and $\mathcal{C} := \{k : k \in \{3,5,7,\cdots\}\}$

$$\begin{aligned}
& R_{achievable} + \frac{1}{2}\log 5 \\
&\overset{(d)}{=} \max_{\mathcal{B}} \min_{\mathcal{C}} \left(\left[\frac{1}{2}\log\left(\frac{P'_1}{N_2}\right)\right]^+, \left[\frac{1}{2}\log\left(\frac{P'_k}{N_{k-1}}\right)\right]^+, \right. \\
&\quad \left. \left[\frac{1}{2}\log\left(\frac{P'_k}{N_{k+1}}\right)\right]^+ \right) + \frac{1}{2}\log 5 \qquad (23) \\
&= \max_{\mathcal{B}} \max \left(\min_{\mathcal{C}} \left(\frac{1}{2}\log\left(\frac{5P'_1}{N_2}\right), \frac{1}{2}\log\left(\frac{5P'_k}{N_{k-1}}\right), \right. \right. \\
&\quad \left. \left. \frac{1}{2}\log\left(\frac{5P'_k}{N_{k+1}}\right) \right), \frac{1}{2}\log 5 \right) \qquad (24) \\
&\overset{(e)}{\geq} \max \left(\min_{\mathcal{C}} \left(\frac{1}{2}\log\left(\frac{5P^\star_1}{N_2}\right), \frac{1}{2}\log\left(\frac{5P^\star_k}{N_{k-1}}\right), \right. \right. \\
&\quad \left. \left. \frac{1}{2}\log\left(\frac{5P^\star_k}{N_{k+1}}\right) \right), \frac{1}{2}\log 5 \right) \qquad (25) \\
&\overset{(f)}{\geq} \min_{\mathcal{C}} \left(C\left(\frac{P_1}{N_2}\right), C\left(\frac{P_k}{N_{k-1}}\right), C\left(\frac{P_k}{N_{k+1}}\right) \right) \qquad (26) \\
&\geq R_{outer}. \qquad (27)
\end{aligned}$$

The first equality (d) follows from an assumption that WLOG, one of $\left[\frac{1}{2}\log\left(\frac{P'_1}{N_2}\right)\right]^+$, $\left[\frac{1}{2}\log\left(\frac{P'_k}{N_{k-1}}\right)\right]^+$, or $\left[\frac{1}{2}\log\left(\frac{P'_k}{N_{k+1}}\right)\right]^+$ for an odd number k: $k = \{3,5,7,\dots\}$ is the tightest constraint (else it is an even constraint that is the tightest, which is symmetric). The first inequality (e) follows since the rates achieved by the optimized powers must be larger than those achieved by one particular strategy that meets the constraints – in this case the strategy that yields the $P^\star_i$ which we construct in Appendix C. Inequality (f) follows from the fact that $4P^\star_i \geq P_i$, as also shown in Appendix C. Finally, we bound $\frac{1}{2}\log\left(\frac{5P^\star_1}{N_2}\right)$ with $\frac{1}{2}\log\left(1+\frac{P_1}{N_2}\right)$ as follows: If $\frac{P_1}{N_2} \geq 4$, it follows that $\frac{P^\star_1}{N_2} \geq 1$, and hence

$$\begin{aligned}
\frac{1}{2}\log\left(\frac{5P^\star_1}{N_2}\right) &= \frac{1}{2}\log\left(\frac{P^*_1}{N_2} + \frac{4P^*_1}{N_2}\right) \\
&\geq \frac{1}{2}\log\left(\frac{P^*_1}{N_2} + \frac{P_1}{N_2}\right) \\
&\geq \frac{1}{2}\log\left(1 + \frac{P_1}{N_2}\right).
\end{aligned}$$

Otherwise, $\frac{1}{2}\log\left(1+\frac{P_1}{N_2}\right) < \frac{1}{2}\log 5$. Similarly, we may bound $\frac{1}{2}\log\left(\frac{5P^\star_k}{N_{k-1}}\right)$ and $\frac{1}{2}\log\left(\frac{5P^\star_k}{N_{k+1}}\right)$ with $\frac{1}{2}\log\left(1+\frac{P_k}{N_{k-1}}\right)$ and $\frac{1}{2}\log\left(1+\frac{P_k}{N_{k+1}}\right)$ for all $k = \{3,5,7,\dots\}$ respectively. ∎

Remark 3: We note that our lattice-based achievability scheme for the full-duplex channel achieves to within a constant gap of the outer bound, which corresponds to two parallel one-way line networks. This means that even in the presence of multiple relays, the usage of full-duplex nodes allows one to essentially "double" the capacity with respect to time-shared half-duplex links.

Remark 4: We also emphasize that our gap to capacity is independent of the number of nodes or relays in the line network. Applying noisy network coding (without any optimization / tailoring) would result in a gap of $0.63N$ [26, Theorem 4], where N is the number of nodes in this line network.

VII. Extension to Half-duplex nodes

The proposed lattice coding scheme may alternatively be tailored to channels with half-duplex nodes, i.e. in which a node may either transmit or receive at a given time but not both. The scheme is illustrated in Figure 4, where we see that the proof generally mimics the full-duplex case (Theorem 7, Lemma 8 and Theorem 9), but that each phase (or block) in the full-duplex case is divided into two phases / blocks in the half-duplex case, as nodes may not transmit and receive at the same time. Thus, one may achieve half the rates of the full duplex case, i.e.

$$R_a, R_b < R_{half-duplex} = \frac{1}{2} R_{achievable},$$

where $R_{achievable}$ is expressed in Theorem 9. This is as expected: the full-duplex version doubles the rate of the half-duplex version.

VIII. Conclusion

We proposed a lattice coding scheme for the full-duplex AWGN two-way two-relay line network ($1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$) which achieves within $\frac{1}{2}\log 3$ bit/Hz/s from the symmetric rate capacity. This scheme was further tailored to half-duplex nodes, and to two-way line networks with an arbitrary number

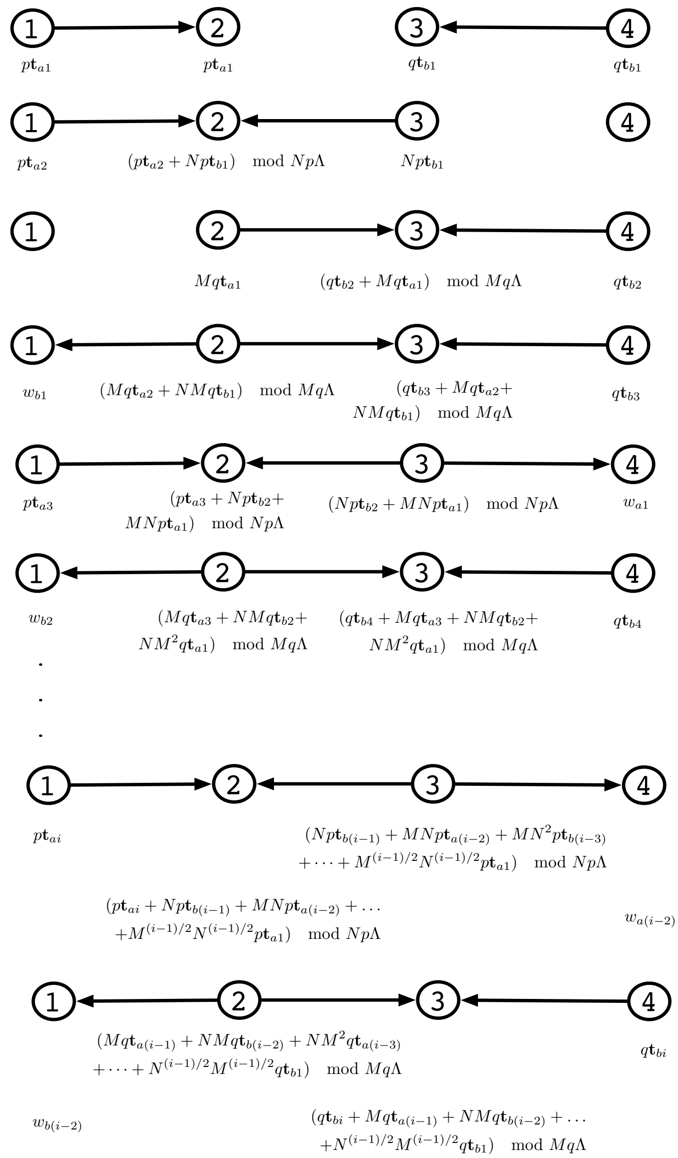

Fig. 4. Half-duplex case.

of relays, where a constant gap $\frac{1}{2}\log 5$ bit/Hz/s of the symmetric capacity is shown to be achievable, regardless of the number of relay nodes. In our scheme, each relay decodes a sum of codewords as all transmitted signals are properly chosen lattice codewords, performs a Folding and Scaling operation that folds the decoded lattice point to another so as to fully utilize the relay transmit power, and broadcasts a scaled version of this folded lattice codeword. All decoders are lattice decoders and only a single nested lattice codebook pair is needed in our scheme. Note that this work results in a symmetric rate region which means the rates of both directions have to satisfy the constraints of all the links. One interesting open question is whether one can derive an achievability scheme which would result in an asymmetric rate region (where rates of each direction are only constrained by links for that direction and the two directions would thus not interfere), and whether this would result in better rates. Overall, we have shown that, just as full duplex operation in a point-to-point channel allows one to transmit in two directions simultaneously without any rate penalty (i.e. two parallel one-way links, or doubling of the capacity), the same may be approximately (to within a constant gap) concluded for general two-way line networks.

Appendix

A. *Multi-phase block Markov achievability strategy for Lemma 8*

The achievable rate region for Lemma 8 is the same as that for Theorem 7; the achievability strategy is essentially the same, with slight variations on the re-scaling to meet the different power constraints. The details of who transmits and decodes what in each phase is outlined in Figure 5.

B. *A choice of $P_i^\star$ and proof that $2P_i^\star \geq P_i$ ($i \in \{1,3\}$) in Theorem 9*

WLOG, we assume $P_3 \geq P_1$. Then, $m^2 \leq \frac{P_3}{P_1} \leq (m+1)^2$ for some integer $m \in \mathbb{Z}^+$. Consider the following strategy for choosing $P_i^\star$ such that $\frac{P_3^\star}{P_1^\star}$ is a non-zero integer squared or the inverse of an integer squared: If $m^2 \leq \frac{P_3}{P_1} \leq m(m+1)$, we choose $P_3^\star = m^2 P_1$ and $P_1^\star = P_1$. Then $\frac{P_3^\star}{P_3} = \frac{m^2 P_1}{P_3} \geq \frac{m^2 P_1}{m(m+1)P_1} \geq \frac{1}{2}$. Thus, $2P_3^\star \geq P_3$ and $P_1^\star = P_1$. Otherwise if $m(m+1) \leq \frac{P_3}{P_1} \leq (m+1)^2$, we choose $P_1^\star = \frac{1}{(m+1)^2}P_3$ and $P_3^\star = P_3$. Then $\frac{P_1^\star}{P_1} = \frac{P_3}{(m+1)^2 P_1} \geq \frac{m(m+1)P_1}{(m+1)^2 P_1} \geq \frac{1}{2}$. Thus $2P_1^\star \geq P_1$ and $P_3^\star = P_3$. In general, this strategy ensures that $2P_i^\star \geq P_i$.

C. *A choice of $P_i^\star$ and proof that $4P_i^\star \geq P_i$ ($i \in \{1,3,5,\ldots\}$) in Theorem 10*

Consider the following strategy for choosing $P_i^\star$ such that $\frac{P_i^\star}{P_{i+2}^\star}$ for all $i \in \{1,3,5,\ldots\}$ are non-zero integer squared or the inverse of an integer squared: We always choose $P_{i+2}^\star$ according to $P_i^\star$. That is we first choose $P_1^\star = P_1$ then set $P_3^\star$, $P_5^\star$, $P_7^\star$ $\ldots$ sequentially. If $P_{i+2} \geq P_i^\star$ and then $m^2 \leq \frac{P_{i+2}}{P_i^\star} \leq (m+1)^2$ for some integer $m \in \mathbb{Z}^+$, we choose $P_{i+2}^\star = m^2 P_i^\star$. Then $\frac{P_{i+2}^\star}{P_{i+2}} \geq \frac{m^2 P_i^\star}{(m+1)^2 P_i^\star} \geq \frac{1}{4}$ and so $4P_{i+2}^\star \geq P_{i+2}$. If $P_{i+2} \leq P_i^\star$ and then $\frac{1}{(n+1)^2} \leq \frac{P_{i+2}}{P_i^\star} \leq \frac{1}{n^2}$ for some integer $n \in \mathbb{Z}^+$, we choose $P_{i+2}^\star = \frac{P_i^\star}{(n+1)^2}$. Then $\frac{P_{i+2}^\star}{P_{i+2}} \geq \frac{n^2 P_i^\star}{(n+1)^2 P_i^\star} \geq \frac{1}{4}$, and again $4P_{i+2}^* \geq P_{i+2}$. Thus, for this strategy, one has $4P_i^\star \geq P_i$ for $i \in \{1,3,5,\ldots\}$. Notice that we need only construct powers for the odd indices since without loss of generality one of these is the tightest constraint.

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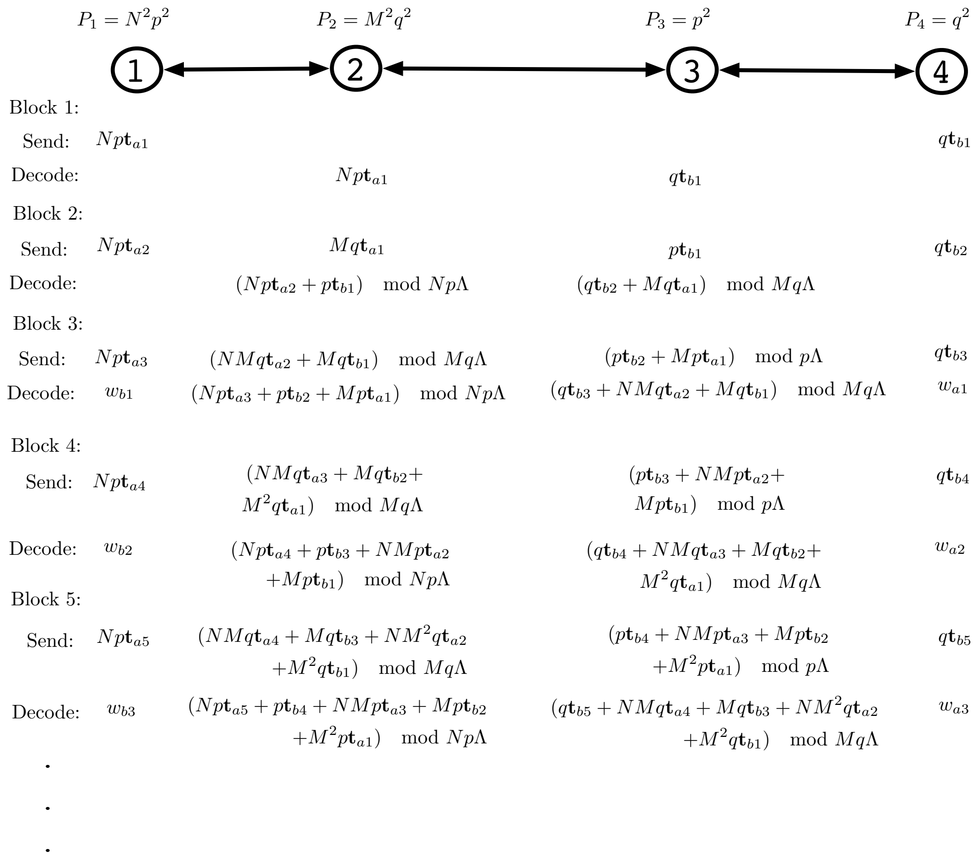

Fig. 5. Multi-phase block Markov achievability strategy for Lemma 8.

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