Radar Waveform Design with The Two Step Mutual Information

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***Abstract*—Waveform design over two consecutive transmission time instants or epochs is the focus of this paper. The waveforms are designed to maximize the mutual information (MI) over the two time epochs. For a single epoch, maximizing the MI is equivalent to waterfilling. In this paper it is shown that for two epochs, the designed waveforms must satisfy two criteria. The first is waterfilling, and the other is that the waveforms must place energies in those spectral bands which offer maximum uncorrelated target scattering responses. The interplay of these two criteria causes the waveforms to be distinct from those designed by the single-epoch waterfilling-based systems. Comparisons of the waveforms designed over one epoch, and over two epochs are shown through simulations.**

I. Introduction

Radars operate by utilizing multiple transmission times or epochs [1]. Traditionally, radar waveforms have been selected from a limited set of options. However, the recent cognitive / fully adaptive radar paradigm [2], [3] allows one to both carefully design and select the waveforms designed in either a pulse-by-pulse or coherent processing interval basis. It may thus be desirable to design waveforms which consider the target response over these multiple epochs jointly. In this work, we make progress in the joint design of waveforms over multiple epochs by first considering two transmission epochs.[1]

We propose to jointly design and transmit the two waveforms that maximize the mutual information [4] over not one but two consecutive time epochs. When the target statistics are available fast enough, the epochs may correspond to two consecutive pulse repetition intervals for example in a pulse-doppler framework. When the target statistics are predicted or obtained infrequently, then the epochs correspond to two consecutive coherent processing intervals for the same framework.

We consider a single extended target, and assume that its impulse responses at the two epochs are random and correlated. Noise plus interference is modeled stochastically, and are assumed to be independent from the target responses. Normality is assumed for tractability. The waveform design is formulated as an optimization problem by maximizing the two step mutual information. In the discrete time domain, the solution to this problem is not straightforward. Therefore, we reformulate the optimization problem in the spectral domain, which is solved numerically. The results in this paper demonstrate that the waveforms designed by the maximizing the two step MI, have to not only "waterfill" over a single epoch, where the interpretation is that the waveform places energy in spectral nulls of the ratio of target response to noise and interference power spectral densities, but also have to balance this with transmitting energy at frequencies where the target scattering responses are maximally independent /uncorrelated over the two epochs.

Literature: Bell's seminal work on waveform design by maximizing the one step (single epoch) MI resulted in a simple waterfilling interpretation of the designed waveform [5], [6]. Waveform design from a purely detection perspective, considering the signal dependent interference (clutter) was the subject of [7]. Surprisingly, after algebraic simplifications, the author shows that this is similar to waterfilling. In [8], the waveform design was addressed by maximizing the one step MI, but by considering signal dependent interference, and led to a solution similar in spirit to that of waterfilling. In [9], [10], waveform scheduling but not waveform design was addressed using information theoretic principles. Specifically it was argued that *directed information* (DI) [11], [12] is a suitable metric when *feedback* is considered in the overall radar transmission-reception scheme [9], [10]. Waveform design using DI as a metric is desired for a closed loop architecture such as a fully adaptive radar, however it is not the immediate focus in this paper for the following reasons. As a first step, the immediate emphasis is to gain a formative understanding on how the two step (or multi step) MI would design the waveforms over the two epochs. Moreover, for the model assumed in this paper maximizing the MI or maximizing the DI are identical. Equivalence of the two step DI and two step MI was also noted for TR channels [13], [14]. Notwithstanding this equivalence, the channel model in [13], [14] is completely different from the one assumed here.

Organization: The paper is organized as follows, in Section II, the model and assumptions are presented, and the two step mutual information in the discrete temporal domain, and its equivalent in the spectral domain is introduced. Waveform design by maximizing this two step MI is treated in Section III. Simulations are presented in Section IV, and conclusions are drawn in Section V.

[1]Extending this work to more than two epochs is fairly straightforward but notationally much more involved.

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II. MODEL

Consider a single complex target consisting of many point scatterers and whose spatial extent possibly spans multiple range cells. Denote $\mathbf{s_1} := [s_1(0), s_1(1), \ldots, s_1(N-1)]^T \in \mathbb{C}^N$ as the waveform transmitted in the first instant, and similarly $\mathbf{s_2} := [s_2(0), s_2(1), \ldots, s_2(N-1)]^T \in \mathbb{C}^N$ is the waveform transmitted in the second instant. The radar returns from the two waveform transmissions are modeled as,

$$\begin{aligned} \mathbf{y_1} &= \mathbf{S_1}\boldsymbol{\alpha_1} + \mathbf{v_1} \\ \mathbf{y_2} &= \mathbf{S_1}\boldsymbol{\alpha_2} + \mathbf{v_2}. \end{aligned} \tag{1}$$

In (1), $\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}$ are the K-dimensional target impulse response vectors in the first and second instants, respectively. The matrices, $\mathbf{S_i} \in \mathbb{C}^{N\times K}, i = 1, 2$ are the respective convolution matrices in the first and second instants, and are comprised of the samples in $\mathbf{s_i}, i = 1, 2$. The noise plus interference in the returns are given by $\mathbf{v_1}$ and $\mathbf{v_2}$ in the two instants, respectively. Signal dependent clutter is ignored for ease of exposition in our analysis, but can be incorporated into the analysis.

Statistical Assumptions We will assume without loss of generality (w.l.o.g.) that the noise plus interference are uncorrelated in the two instants, i.e.

$$\mathbb{E}\{\mathbf{v_1}\mathbf{v_2}^H\} = \mathbf{0}$$

where $\mathbb{E}\{\cdot\}$ is the statistical expectation operator, and $\mathbf{0}$ is an $N \times N$ zero matrix. This allows us to focus on the effect of designing the two waveforms based on the predicted target statistics. In general however, if noise and interference are also correlated, this would further impact the design. However, even our initial simplification turns out to be challenging analytically. Further, for analytical tractability in the subsequent information theoretic analysis we will assume that the target impulse responses are complex circular Gaussian distributed and w.l.o.g. zero mean. Similarly, the noise vectors are also complex circular Gaussian distributed, w.l.o.g. zero mean, and independent of the target responses. The covariance matrices, then, completely specify their joint as well individual distributions, expressed as

$$\begin{aligned} \mathbf{C}_{\boldsymbol{\alpha}} &= \mathrm{cov}\{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H\} = \begin{bmatrix} \mathbf{C}_{\boldsymbol{\alpha_1}} & \mathbf{C}_{\boldsymbol{\alpha_{12}}} \\ \mathbf{C}^H_{\boldsymbol{\alpha_{12}}} & \mathbf{C}_{\boldsymbol{\alpha_2}} \end{bmatrix} \\ \mathbf{C}_{\mathbf{v}} &= \mathrm{cov}\{\mathbf{v}, \mathbf{v}^H\} = \begin{bmatrix} \mathbf{C}_{\mathbf{v_1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{\mathbf{v_2}} \end{bmatrix} \\ \boldsymbol{\alpha} &= [\boldsymbol{\alpha_1}^T, \boldsymbol{\alpha_2}^T]^T, \mathbf{v} = [\mathbf{v_1}^T, \mathbf{v_2}^T]^T \end{aligned} \tag{2}$$

where $\mathrm{cov}\{\cdot\}$ is the covariance operator.

A. Mutual Information in the temporal domain

For the model in (1), the mutual information $I(\boldsymbol{\alpha}; \mathbf{y}) = \mathbb{E}_{p(\boldsymbol{\alpha},\mathbf{y})}\{\ln(\frac{p(\boldsymbol{\alpha},\mathbf{y})}{p(\boldsymbol{\alpha})p(\mathbf{y})})\}$ is sought, which is given by

$$\begin{aligned} I(\boldsymbol{\alpha}; \mathbf{y}) &:= h(\mathbf{y}) - h(\mathbf{y}|\boldsymbol{\alpha}) \\ &= h(\boldsymbol{\alpha}) - h(\boldsymbol{\alpha}|\mathbf{y}) \\ \mathbf{y} &: = [\mathbf{y_1}^T, \mathbf{y_2}^T]^T \end{aligned} \tag{3}$$

where $h(\cdot|\cdot)$ is defined as the conditional differential entropy. The following standard definitions are also recalled [4]

$$\begin{aligned} h(\boldsymbol{\alpha}) &: = -\mathbb{E}_{p(\boldsymbol{\alpha})}\{\ln(p(\boldsymbol{\alpha})\} \\ h(\boldsymbol{\alpha}|\mathbf{y}) &:= -\mathbb{E}_{p(\boldsymbol{\alpha},\mathbf{y})}\{\ln(p(\boldsymbol{\alpha}|\mathbf{y})\} \end{aligned} \tag{4}$$

where $p(\cdot)$ and $p(\cdot|\cdot)$ denote the pdf and conditional pdf, respectively. To evaluate the MI, the following fact is useful.

Fact 1. *If $\mathbf{x}$ is a multivariate Gaussian random vector with an arbitrary mean vector and a covariance matrix, $\mathbf{C_x}$, then the differential entropy (nats), $h(\mathbf{x}) = \ln\det(\mathbf{C_x}) + \eta$, where η is some constant.*

From standard chain rules of MI and differential entropy, the MI in (3) could be rewritten using other MI terms as well as other conditional MI terms, $I(\cdot|\cdot)$ (see [4]) as

$$I(\boldsymbol{\alpha}; \mathbf{y}) = I(\boldsymbol{\alpha_1}; \mathbf{y_1}) + I(\boldsymbol{\alpha_2}; \mathbf{y_2}) \tag{5a}$$
$$+ I(\mathbf{y_1}, \boldsymbol{\alpha_1}; \boldsymbol{\alpha_2}|\mathbf{y_2}) + I(\mathbf{y_2}; \boldsymbol{\alpha_1}|\mathbf{y_1}) \tag{5b}$$
$$- I(\boldsymbol{\alpha_1}; \boldsymbol{\alpha_2}). \tag{5c}$$

We will revisit (5), specifically (5a) since it has significance in the later discussions.

B. Mutual Information in the spectral domain

The MI in (3) is also evaluated in the spectral domain. This representation is most useful in interpreting the designed waveform in the spectral domain. Assume that the radar operates with a bandwidth denoted by W. Let us divide the bandwidth into P consecutive bands each of infinitesimal width denoted by δf. Denote the center frequency of the p-th band as $f_p, p = 1, \ldots, P$. The idea now is to decompose the components of the radar returns in these spectral bands. Define $S_{1p}(f) = S_1(f_p)r_p(f)$, and $S_{2p}(f) = S_2(f_p)r_p(f)$. Here, $S_i(f), i = 1, 2$ is the equivalent spectral representation of the waveform vectors $\mathbf{s_i}, i = 1, 2$. The indicator function is denoted as $\mathbb{1}_{[\cdot]}$, and $r_p(f) := \mathbb{1}_{[f_p-\delta f/2 \le f \le f_p+\delta f/2]}$. Similarly define $V^i_p(f) := V_i(f_p)r_p(f), i = 1, 2$ and the energy spectral variance (ESV) $\Gamma^i_p(f) = \Gamma_i(f_p)r_p(f), i = 1, 2$, where, $V_i(f)$, and $\Gamma_i(f)$ denote the PSD and ESV corresponding to the random processes comprised of the samples in $\mathbf{v_i}, i = 1, 2$ and in $\boldsymbol{\alpha_i}$, respectively. Similarly, we can define the cross ESV to be $\Gamma^p_{12}(f) = \Gamma_{12}(f_p)r_p(f)$ of the target impulse responses at the two instants and in the p-th subband, respectively.

The MI in the p-th band can now be written as

$$T\delta f \ln\left(1 + \frac{\Phi_1(f_p, |S_{1p}(f_p)|^2, |S_{2p}(f_p)|^2)}{T^2\Phi_2(f_p)}\right)$$

where

$$\begin{aligned} \Phi_1(f_p, |S_{1p}(f_p)|^2, |S_{2p}(f_p)|^2) =& |S_{1p}(f_p)|^2|S_{2p}(f_p)|^2\Gamma^1_p(f)\Gamma^2_p(f) \\ &+ T|S_{1p}(f_p)|^2\Gamma^1_p(f)V^2_p(f) \\ &+ T|S_{2p}(f_p)|^2\Gamma^2_p(f)V^1_p(f) \\ &- |S_{1p}(f_p)|^2|S_{2p}(f_p)|^2\Gamma^p_{12}(f) \\ \Phi_2(f_p) =& V^1_p(f)V^2_p(f) \end{aligned}$$

Considering any two non-overlapping bands, the total MI in these bands is the sum of the individual MI. Hence in the limiting case, the total MI over the radar bandwidth is expressed as

$$\begin{aligned} & I_s(\boldsymbol{\alpha};\mathbf{y}) \\ &= \sum_p \lim_{\delta f \to 0} T\delta f \ln\left(1 + \frac{\Phi_1(f_p, |S_1(f_p)|^2, |S_2(f_p)|^2)}{T^2\Phi_2(f_p)}\right) \\ &= T\int\limits_W \ln\left(1 + \frac{\Phi_1(f, |S_1(f_p)|^2, |S_2(f_p)|^2)}{T^2\Phi_2(f)}\right) df \end{aligned} \tag{6}$$

where T is measured in seconds and is the actual duration of transmission plus the duration of the target impulses responses as well as the impulse responses of the receiver passband filters[2], see also [5]. Further $I_s(\cdot;\cdot)$ is the spectral domain equivalent of the MI defined in (3) and obtained as in (6).

III. Waveform Design

We are interested in designing waveforms over two consecutive epochs by maximizing the two-step MI. This differs from prior work in [9], [10], where waveform scheduling from a library of waveforms according to a certain metric, rather than design, was investigated.

A. Maximizing the MI in the temporal domain

Consider the constrained optimization problem,

$$\begin{aligned} \max_{\mathbf{s_1},\mathbf{s_2}} \quad & I(\boldsymbol{\alpha};\mathbf{y}) \\ \text{s. t.} \quad & ||\mathbf{s_1}||^2 \le P_o \\ & ||\mathbf{s_2}||^2 \le P_o \end{aligned} \tag{7}$$

where P_o is a physical constant, which corresponds to the maximum energy tolerance by the hardware, and is assumed to be identical in both the transmission instants. Using (3) in (7), we have

$$\begin{aligned} \min_{\mathbf{s_1},\mathbf{s_2}} \quad & h(\boldsymbol{\alpha}|\mathbf{y}) \\ \text{s. t.} \quad & ||\mathbf{s_1}||^2 \le P_o \\ & ||\mathbf{s_2}||^2 \le P_o \end{aligned} \tag{8}$$

Unfortunately, solving (8) is not straightforward. Nevertheless we can relate the waveforms designed by maximizing the MI to more commonly used estimation metrics such as the Bayesian mean square error (BMSE) [9], [10]. From (1) and normality assumptions, it can be shown that $h(\boldsymbol{\alpha}|\mathbf{y}) = \ln\det\{(\mathbf{C}_{\boldsymbol{\alpha}}^{-1} + \mathbf{H}^H\mathbf{C}_{\mathbf{v}}^{-1}\mathbf{H})^{-1}\} + \text{const.}$, where $\mathbf{H} := \begin{bmatrix} \mathbf{S_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S_2} \end{bmatrix}$. It is now readily recognized that the Bayesian mean square error (BMSE) of $\boldsymbol{\alpha}$ given $\mathbf{y}$, is $\text{BMSE}(\boldsymbol{\alpha}|\mathbf{y}) = (\mathbf{C}_{\boldsymbol{\alpha}}^{-1} + \mathbf{H}^H\mathbf{C}_{\mathbf{v}}^{-1}\mathbf{H})^{-1}$, see also [9], [10], [15]. In other words, under normality assumptions and since $\ln(\cdot)$ is strictly monotonic, the waveforms which maximize the MI, $I(\boldsymbol{\alpha};\mathbf{y})$ also minimize the determinant of the BMSE [9], [10].

[2]Passband w.r.t the operating radar bandwidth

B. Maximizing the MI in the spectral domain

Using (6) in (7), the optimization is now rewritten as

$$\begin{aligned} \max_{|S_1(f)|^2, |S_2(f)|^2} \quad & T\int\limits_W \ln\left(1 + \frac{\Phi_1(f, |S_1(f)|^2, |S_2(f)|^2)}{T^2\Phi_2(f)}\right) df \\ \text{s. t.} \quad & \int\limits_W |S_1(f)|^2\, df \le P_o \\ & \int\limits_W |S_2(f)|^2\, df \le P_o. \end{aligned} \tag{9}$$

It is noted that unlike (7)-(8), the optimization in (9) is w.r.t the magnitude squared spectrum of the waveforms. In other words, the phase of the waveforms in the spectral domain cannot be determined uniquely. Such insights are not readily gleaned from (7)-(8), since they are not amenable to further simplification.

The (aggregate, i.e. for the entire bandwidth) Lagrangian for the optimization in (9) is given by,

$$\begin{aligned} & \mathcal{L}(|S_1(f)|^2, |S_2(f)|^2, \lambda_1, \lambda_2) \\ &= \int\limits_W \mathcal{L}_f(|S_1(f)|^2, |S_2(f)|^2, \lambda_1, \lambda_2) df - (\lambda_1 + \lambda_2)P_o \end{aligned} \tag{10}$$

where,

$$\begin{aligned} & \mathcal{L}_f(|S_1(f)|^2, |S_2(f)|^2, \lambda_1, \lambda_2) \\ &= T\ln\left(1 + \frac{\Phi_1(f, |S_1(f)|^2, |S_2(f)|^2)}{T^2\Phi_2(f)}\right) \\ &+ \lambda_1|S_1(f)|^2 + \lambda_2|S_2(f)|^2. \end{aligned}$$

The first order gradient conditions are

$$\frac{d\mathcal{L}_f(|S_1(f)|^2, |S_2(f)|^2, \lambda_1, \lambda_2)}{d|S_i(f)|^2} = 0, i = 1, 2. \tag{11}$$

We may then numerically solve for $|S_i(f)|^2, i = 1, 2$ using (11) along with the rest of the KKTs, i.e., complementary slackness, constraint qualifications, and $\lambda_i \ge 0$. This procedure is straightforward but nevertheless numerically involved, see also [5] on how this is done for the single step MI waveform design.

C. Scalar special case:

The scalar equivalent of (1) allows for a mathematically simpler interpretation of the waveform design. Define the scalar model as

$$\begin{aligned} y_i &= s_i\alpha_i + n_i, i = 1, 2 \\ \text{cov}\{\mathbf{y}, \mathbf{y}^H\} &= \begin{bmatrix} |s_1|^2\sigma_1^2 + \sigma^2 & s_1\sigma_{12}s_2^* \\ s_1^*\sigma_{12}^*s_2 & |s_2|^2\sigma_2^2 + \sigma^2 \end{bmatrix} \\ \text{cov}\{\boldsymbol{\alpha}, \boldsymbol{\alpha}^H\} &= \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12}^* & \sigma_2^2 \end{bmatrix} \end{aligned} \tag{12}$$

where $\mathbf{y} = [y_1, y_2]^T, \boldsymbol{\alpha} = [\alpha_1, \alpha_2]^T$, and without of loss of generality we assumed that the noise has identical variance equal to σ^2 for both epochs. In (12) the variances of α_i are σ_i^2 and the corresponding correlation coefficient is proportional to

$|\sigma_{12}|^2$. The MI for the scalar model in (12) is readily derived, and given below.

$$I(\boldsymbol{\alpha};\mathbf{y}) = \ln\left(1+\overbrace{\frac{|s_1|^2\sigma_1^2+|s_2|^2\sigma_2^2}{\sigma^2}}^{\textbf{waterfilling}}+\underbrace{\frac{|s_1|^2|s_2|^2(\sigma_1^2\sigma_2^2-|\sigma_{12}|^2)}{\sigma^4}}_{\textbf{second criteria}}\right)$$
$$= \ln\left(1+\frac{|s_1|^2\sigma_1^2+|s_2|^2\sigma_2^2}{\sigma^2} + \frac{|s_1|^2|s_2|^2\sigma_1^2\sigma_2^2\exp(-I(\alpha_1;\alpha_2))}{\sigma^4}\right). \quad (13)$$

From (13), it is clear that the scalar MI consists of two parts. The first is responsible for waterfilling. In fact the waterfilling part was already evident from (5a). The other comprises the part which is maximized when the MI, $I(\alpha_1;\alpha_2)$ is minimized. Further, it is clear that incremental increases in the energy of the waveforms when $I(\alpha_1;\alpha_2)$ is relatively small, leads to substantial increases in this second term. Compare this to the case when $I(\alpha_1;\alpha_2)$ is relatively large, then, similar incremental increases in the waveform's energy do not lead to a large increases in this second term. Extrapolating this directly to the vector channels as in (1), we can state that for maximizing this analogous second term, most benefit is realized by placing energy in those spectral bands where the target scattering responses have the least MI, or, are maximally uncorrelated.

Special case: When $\sigma_{12}=0$, then from (13)

$$I(\mathbf{y};\boldsymbol{\alpha}) = I(y_1,\alpha_1)+I(y_2,\alpha_2).$$

Hence maximizing $I(\mathbf{y};\boldsymbol{\alpha})$ is equivalent to maximizing the individual $I(y_i,\alpha_i), i=1,2$. In other words, waterfilling on each epoch is sufficient. This also directly applies to the vector channel as well, when, $\mathbf{C}_{\boldsymbol{\alpha}_{12}}=\mathbf{0}$.

IV. Simulations

The radar bandwidth, $W=100$ MHz in the simulations. To develop an understanding of how the MI designs the waveforms, we will consider some contrived examples of the target ESVs and noise PSDs.

Example-1 The target ESV in the first and second instants are as well as the noise PSDs in the two instants are seen in Fig. 1 within W. Spectral content outside the radar bandwdith is notched. The target ESVs in the first and second instants are highly frequency selective, and are modeled as sinusoidally varying in the frequency variable. The noise PSDs in both the instants have deep notches at DC and are clearly colored. The cross spectral PSD of the noise is assumed to be zero at all frequencies.

In Fig.2(a), $\Gamma_1(f)\Gamma_2(f)-\Gamma_{12}^2(f)$ is shown over the entire radar bandwidth. Peaks in this function represent frequencies where the target responses at the two epochs are highly uncorrelated, whereas, frequencies where this function assumes low values imply a strong dependence or correlation between the

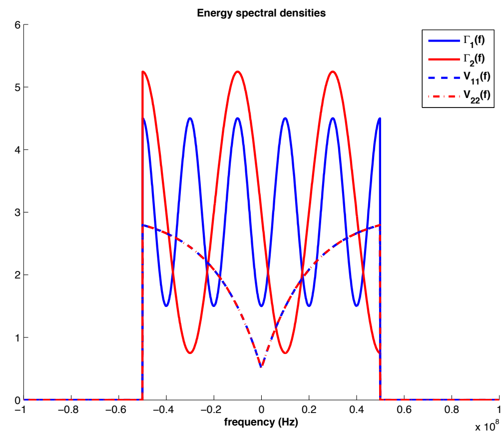
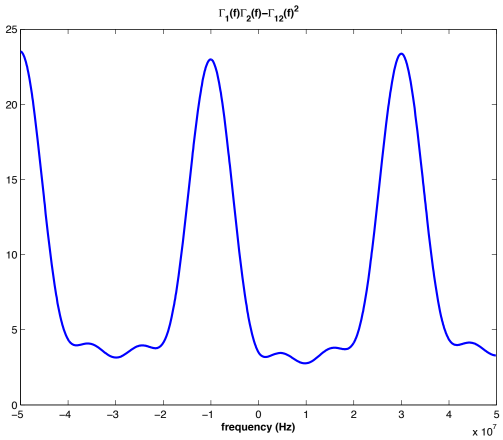


Fig. 1. Noise PSDs and target impulse response ESVs for Example -1

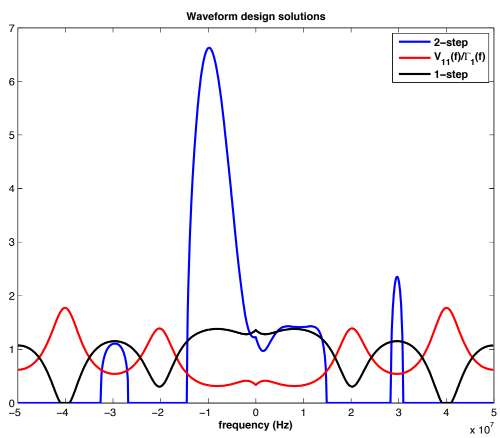
target responses at the two epochs. The designed waveforms (blue) using the two step MI and for the two epochs are shown in Fig.2(b),(c), respectively. For comparison, the waterfilling solution (red) for the two epochs, as well as the criteria used for waterfilling (red) is also shown. We see that the black curves are mostly inverted versions of the red curves in Fig. 2(b)(c). The waveforms designed using waterfilling alone are quite different from those designed using the two step MI. In Fig. 2(b), the two step solution puts sufficient energy at around 30 MHz to just satisfy the waterfilling criteria and then greedily places energy between 0-10 MHz since these frequencies from Fig. 2(a) are uncorrelated for the two epochs. Other insights are drawn in a straightforward manner. Consider Fig. 2(c), the two step solution is more or less identical to the waterfilled solution except for a sharper roll off in the latter than the former, attributed to the shaping by $\Gamma_1(f)\Gamma_2(f)-\Gamma_{12}^2(f)$. It is no surprise that for this figure the waterfiiling and the two-step solution are nearly identical since the waterfilling also places energy where the target responses are maximally uncorrelated.

Example-2 The target ESV in the first epoch is now made Gaussian shaped in frequency, whereas the target ESV in the second epoch is assumed to be frequency selective and sinusoidal w.r.t frequency. The noise PSDs are assumed identical to the ones assumed in example-1. The target ESVs and noise PSDs are seen in Fig. 3.

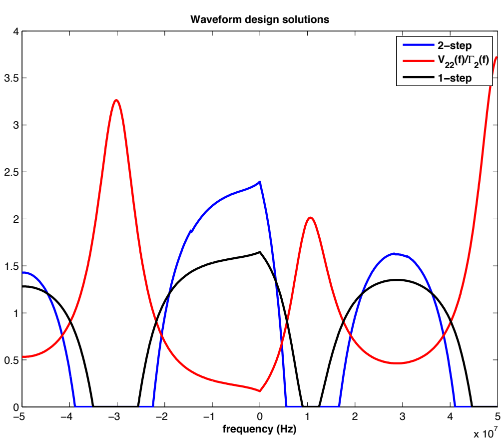
The function $\Gamma_1(f)\Gamma_2(f)-\Gamma_{12}^2(f)$ for this example, is shown over the entire radar bandwidth in Fig. 4(a), and the designed waveforms for the two step and single step MI are shown in Fig. 4(b),(c). For the first slot, the two step MI as seen in Fig. 4(b), places no energy for $|f|\geq 25$ MHz, but rather places most of its energy in bands comprising $|f|\leq 20$ MHz. Moreover, there are two sharply defined peaks in this band which are consistent with the corresponding peaks in Fig. 4(a) in the same frequency range. For the single step MI the waveform, as expected, has a shape which is similar to the inverse of the function $V_{11}(f)/\Gamma_1(f)$. What is surprising is that for frequency range $|f|\geq 30$ MHz the single step MI waveform energy is non -zero, although it is amply evident



(a)



(b)



(c)

Fig. 2. Results for example-1, criteria for waterfiling (red), waterfilling solution (black), and two step MI solution (blue) are shown, and (a) $\Gamma_1(f)\Gamma_2(f) - \Gamma_{12}^2(f)$, b) $|S_1(f)|^2$ c) $|S_2(f)|^2$.

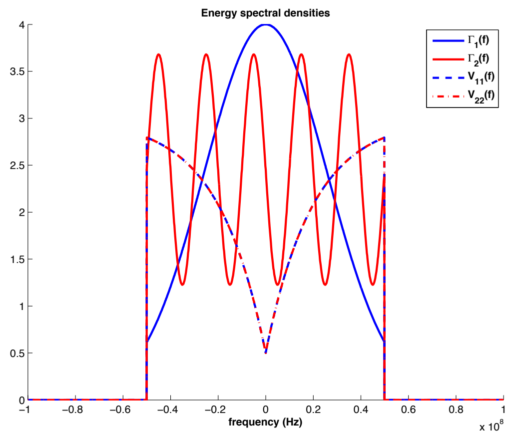


Fig. 3. Noise PSDs and target impulse response ESVs for Example -2

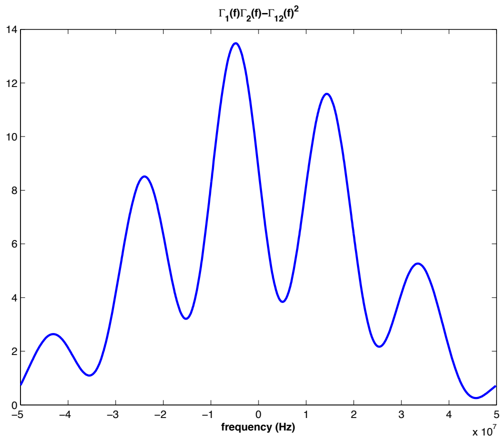
that this energy is being wasted since it is always below the red curve. This behavior is not demonstrated by the two step MI.

For the second slot, the two step MI's waveform as seen in 4(c) has peaks which are consistent with all the five peaks in 4(a). Further it is also clear that it has several nulls in those frequency ranges where the the $V_{22}(f)/\Gamma_2(f)$ is greater than the single step MI's waveform. This is important from an SNR perspective since $V_{ii}(f)/\Gamma_i(f), i = 1, 2$ are the frequency dependent denominators of the frequency dependent SNR defined as $\frac{|S_i(f)|^2}{TV_{ii}(f)/\Gamma_i(f)}$. This reinforces the fact that the waveform designed from the two step MI judiciously places energy in spectral bands, unlike the waveform designed from the single step MI criteria from a frequency dependent SNR perspective.

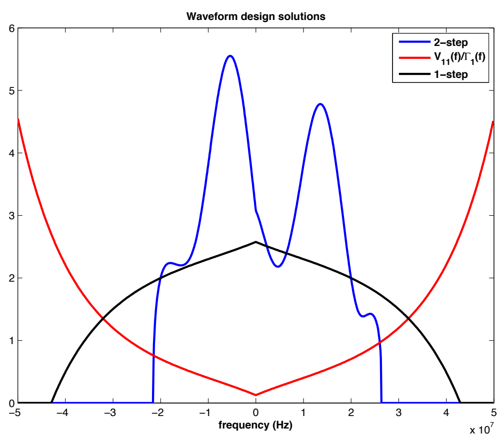
V. Conclusions

In this paper waveform design in radar over two consecutive transmission epochs was treated. The two step MI was used as a design metric. The two step MI channel models were treated both in the temporal domain as well as the spectral domain. The spectral domain representation of the MI was then used in the subsequent analysis. The scalar channel was also analyzed to obtain an intuitive analytical understanding of the waveform design technique.

It was then shown that for the two epochs, the designed waveforms using the two step MI design metric must satisfy two criteria. The first is spectral domain waterfilling, and the other is that the waveforms must place energies in those spectral bands which offer maximum un-correlated target scattering responses. The waveforms designed from two step MI were then compared to the waveforms designed from the single step MI via simulations. From a frequency dependent SNR argument, the simulations indicated that waveforms designed from the two step MI judiciously places energy in spectral bands and simultaneously placed nulls in some other bands. This was unlike the waveforms designed from the single step MI criteria which tend to, at least partially, expend energy in some spectral bands with seemingly little to no benefit.



(a)



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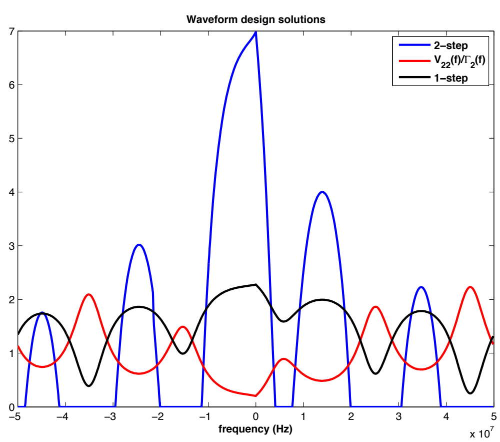

(c)

Fig. 4. Results for example-2 criteria for waterfiling (red), waterfilling solution (black), and two step MI solution (blue) are shown, and (a) $\Gamma_1(f)\Gamma_2(f) - \Gamma_{12}^2(f)$, b) $|S_1(f)|^2$ c) $|S_2(f)|^2$

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