CS202 Fall 2012
Lecture 9 – 9/27

Probability Flashback

Prof. Tanya Berger-Wolf
http://www.cs.uic.edu/bin/view/CS202/WebHome

Bernoulli Trials

A coin is tossed 8 times. What is the probability of exactly 3 heads in the 8 tosses?

THHTTHTT is a tossing sequence…

How many ways of choosing 3 positions for the heads?

$C(8,3)$

What is the probability of a particular sequence?

$.5^8$

In general: The probability of exactly k successes in n independent Bernoulli trials with probability of success p, is

$C(n,k)p^k(1-p)^{n-k}$

Bernoulli Trials

A game of Jewel Quest is played 5 times. You clear the board 70% of the time. What is the probability that you win a majority of the 5 games?

Sanity check: What is the probability the the result is WWLLW?

$.7^3.3^2$ Assumes independent trials

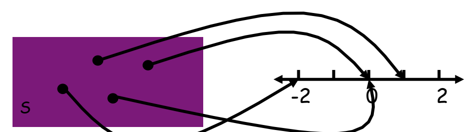
In general: The probability of exactly k successes in n independent Bernoulli trials with probability of success p, is

$C(n,k)p^k(1-p)^{n-k}$

$C(5,3)0.7^3 0.3^2 + C(5,4)0.7^4 0.3^1 + C(5,5)0.7^5 0.3^0$

Random Variables

For a given sample space S, a *random variable* is any real valued function on S.



Suppose our experiment is a roll of 2 dice. S is set of pairs.

- X = sum of two dice. X((2,3)) = 5
- Y = difference between two dice. Y((2,3)) = 1
- Z = max of two dice. Z((2,3)) = 3

Random Variables

A probability distribution on a r.v. X is just an allocation of the total probability, 1, over the possible values of X.

How many movies have you watched in the last week?

a) 0
b) 1
c) 2
d) 3
e) 4

Picture gives a probability distribution!

The chart gives likelihood that a randomly selected student watched each of the particular numbers of movies.

Random Variables

Example:

Suppose we are playing a game with cards labeled 1 to 20, and we draw 3 cards. We bet that the maximum card has value 17 or greater. What's the probability we win the bet?

Let r.v. X denote the maximum card value. The possible values for X are 3, 4, 5, …, 20.

i	3	4	5	6	7	8	9	…	20
Pr(X = i)	?	?	?	?	?	?	?		?

Filling in this box would be a pain. We look for a general formula.

Random Variables

X is value of the highest card among the 3 selected. 20 cards are labeled 1 through 20.

We want Pr(X = i), i = 3,…20.

Denominator first: How many ways are there to select the 3 cards? C(20,3)

a) 20
b) 6840
c) 60
d) 1140
e) I'm not telling.

How many choices are there that result in a max card whose value is i? C(i-1,2)

Pr(X = i) = C(i-1, 2) / C(20,3) These are the table values.

We win the bet if the max card, X is 17 or greater. What's the probability we win? Pr(X = 17) + Pr(X = 18) + Pr(X = 19) + Pr(X = 20) ≈0.51

A Special Random Variable

A *Binomial* random variable X is a random variable that counts the number of successes in a sequence of n independent Bernoulli trials, where the probability of success on each trial is p.

What are the possible values for X? 0,1, 2, …n

We want Pr(X = k), k = 0, 1,…n.

From last time,

$$Pr(X = k) = C(n, k)\, p^k (1-p)^{n-k}$$

A Special Random Variable

Suppose a network consisting of 10 computers crashes completely if there are not at least 8 functioning machines (at most 2 failed machines). Further, suppose that each machine functions with probability 0.9, independently of the others.

What is the probability the network is available?

Recognize that the r.v. of interest is a Binomial r.v.

X = "the number of functioning machines out of 10, where probability of a machine functioning is 0.9."

We want P(X=8) + P(X=9) + P(X=10)
= $C(10,8)(0.9)^8(0.1)^2 + C(10,9)(0.9)^9(0.1)^1 + C(10,10)(0.9)^{10}(0.1)^0$
= 45*0.43046721*0.01 + 10*0.387420489*0.1 +0.3486784401
= 0.1937102445 + 0.387420489 + 0.3486784401
= 0.9298091736

A Special Random Variable

Suppose a network consisting of 10 computers crashes completely if there are not at least 8 functioning machines (at most 2 failed machines). Further, suppose that each machine functions with probability 0.9, independently of the others.

What is the probability the network is **UN**available?

X = "the number of functioning machines out of 10, where probability of a machine functioning is 0.9."

We want P(X=0) + P(X=1) + P(X=2) + … + P(X=7)
= 1 – (P(X=8) + P(X=9) + P(X=10))
= 1 - 0.9298091736 = 0.0701908264

Expected Value

Let X be a discrete r.v. with set of possible values D. The *expected value* of X is:

$$E(X) = \sum_{x \in D} x \cdot \Pr(X = x)$$

Measure of central tendency.

Let X denote your score on the coming midterm. Suppose I assign scores according to the following distribution:

i	55	65	80	90
Pr(X=i)	0.1	0.3	0.4	0.2

Then E(X) = (55)(0.1) + 65(0.3) + (80)(0.4) + 90(0.2) = 75

Expected Value

Let X be a binomial r.v. with parameters n and p.

That is, X is the number of "successes" on n trials where each trial has probability of success p.

Defn of Binomial Distribution.

What is E(X)?

First we need Pr(X = k) $= C(n, k)\, p^k (1-p)^{n-k}$

$$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

Expected Value

$$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$
$$= \sum_{k=0}^{n} \frac{k \cdot n!}{(n-k)!k!} p^k (1-p)^{n-k}$$
$$= \sum_{k=1}^{n} \frac{n!}{(n-k)!(k-1)!} p^k (1-p)^{n-k}$$
$$= np \sum_{k=1}^{n} \frac{n-1!}{(n-k)!(k-1)!} p^{k-1} (1-p)^{n-k}$$
$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}$$
$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^{k} (1-p)^{n-1-k} \quad = np[p + (1-p)]^{n-1}$$
$$= np$$

Expected Value

Let X_i, i= 1,2,…,n, be a sequence of random variables, and suppose we are interested in their sum. The sum is a random variable itself with expectation given by:

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

The proof of this is inductive and algebraic. You can find it almost anywhere – have fun!

Expected Value $\quad E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$

Suppose you all (n) put your cell phones in a pile in the middle of the room, and I return them randomly. What is the expected number of students who receive their own phone back?

Define for i = 1, … n, a random variable:

$$X_i = \begin{cases} 1 & \text{if student i gets the right phone,} \\ 0 & \text{otherwise.} \end{cases}$$

k	0	1
$Pr(X_i=k)$	1-(1/n)	1/n

$E[X_i]$ =
a) 1/n
b) 1/2
c) 1
d) No clue

$E[X_i] = Pr(X_i = 1)$

Expected Value $\quad E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$

Suppose you all (n) put your cell phones in a pile in the middle of the room, and I return them randomly. What is the expected number of students who receive their own phone back?

Define for i = 1, … n, a random variable:

$$X_i = \begin{cases} 1 & \text{if student i gets the right phone,} \\ 0 & \text{otherwise.} \end{cases}$$

Define r.v. $X = X_1 + X_2 + … + X_n$, and we want E[X].

$E[X] = E[X_1 + X_2 + … + X_n]$
$= E[X_1] + E[X_2] + … + E[X_n]$
$= 1/n + 1/n + … + 1/n$
$= 1$

Expected Value $E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$

Suppose there are N couples at a party, and suppose m people get sleepy and leave. What is the expected number of couples left?

Define for i = 1, … N, a random variable:

$$X_i = \begin{cases} 1 & \text{if couple i remains,} \\ 0 & \text{otherwise.} \end{cases}$$

Define r.v. $X = X_1 + X_2 + \ldots + X_n$, and we want E[X].

$E[X] = E[X_1 + X_2 + \ldots + X_n]$

$= E[X_1] + E[X_2] + \ldots + E[X_n]$

So what do we know about X_i?

Expected Value $E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$

Suppose there are N couples at a party, and suppose m people get sleepy and leave. What is the expected number of couples left?

Define for i = 1, … N, a random variable:

$$X_i = \begin{cases} 1 & \text{if couple i remains,} \\ 0 & \text{otherwise.} \end{cases}$$

$E[X_i] = Pr(X_i = 1)$ (# of ways of choosing m from everyone else) / (# of ways of choosing m from all)

$$= \frac{\binom{2N-2}{m}}{\binom{2N}{m}}$$

$E[X_1] + E[X_2] + \ldots + E[X_n]$

$= n \times E[X_1] = (2N-m)(2N-m-1)/2(2N-1)$

Variance

What is difference between the following distributions?

k	0	1	2
P(k)	.1	.8	.1

k	0	1	2
P(k)	.4	.2	.4

Need a measure of spread.

Use the average squared difference from the mean.

Variance

Definition: Let X be a r.v. on a sample space S. The variance of X, denoted by Var(X), is:

$$Var(X) = \sum_{s \in S} (X(s) - E[X])^2 Pr(X = s)$$

Theorem: If X is a r.v. on a sample space S, then

$$Var(X) = E[X^2] - E[X]^2$$

By algebra shown in your book, page 389.

Variance $\mathrm{Var}(X) = E[X^2] - E[X]^2$

What is difference between the following distributions?

k	0	1	2
P(k)	.1	.8	.1

k	0	1	2
P(k)	.4	.2	.4

k	0	1	4
P(k)	.1	.8	.1

k	0	1	4
P(k)	.4	.2	.4

$E[X^2] = (0)(.1)+1(.8)+4(.1) = 1.2$

$\mathrm{Var}(X) = 1.2 - 1 = 0.2$

$E[X^2] = (0)(.4)+1(.2)+4(.4) = 1.8$

$\mathrm{Var}(X) = 1.8 - 1 = 0.8$

Done with Probability…

Summary and overview: why is all this probability important?

We can argue for a long time whether or not *life* is random. But maybe we can agree that, given our limited knowledge, we have to *act* as if it is.

Applications in computer science:

- Reliability of parts
- Algorithm analysis on "random" data
- Randomized algorithms
- User behavior
- Data Mining
- Probably gazillions more…