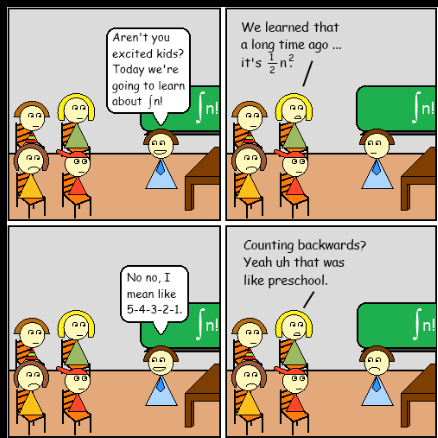

http://spikedmath.com/368.html

CS202 Fall 2012 Lecture 5 – 9/6

Combinatorics: More Counting

Prof. Tanya Berger-Wolf
http://www.cs.uic.edu/bin/view/CS202/WebHome

Announcements

1. Homework 1 due on Thursday

2. Peer tutoring is now available
The tutor schedule is here:
https://docs.google.com/document/pub?id=10dm_onVvXjaHYznx3ypo3fubnQQPq7tawIELVv2aacM

(Also accessible from http://www.cs.uic.edu/Main/UndergraduatePrograms)

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Combinations

- How many 8 bit strings with exactly three 1's?
- How many distinct words are there from MISSISSIPPI?

Suppose a collection consists of n objects of which
n_1 are of type 1 and are indistinguishable from each other
n_2 are of type 2 and are indistinguishable from each other
...
n_k are of type k and are indistinguishable from each other

and $\sum_{i=1}^{k} n_i = n$. Then the number of distinct permutations of the n objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!n_2!n_3!\cdots n_k!}$$

Combinations with Repetitions Allowed

How many multisets of size three are there from {1,2,3,4}?

{1,1,1} {1,1,2} {1,1,3} {1,1,4}
{1,2,2} {1,2,3} {1,2,4}
{1,3,3} {1,3,4} {1,4,4}
{2,2,2} {2,2,3} {2,2,4}
{2,3,3} {2,3,4} {2,4,4}
{3,3,3} {3,3,4} {3,4,4}
{4,4,4}

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Category 1	Category 2	Category 3	Category 4	Result of Selection
x x x				{1, 1, 1}
x		x	x	{1, 3, 4}
	x x		x	{2, 2, 4}

Arranging 3 x and 3 | in the 6 positions. Choose where to put x and put the | in the remaining positions: C(6,3) = 6!/(3!3!) = 20

Combinations with Repetitions Allowed

How many solutions to the equation $x_1 + x_2 + x_3 = 11$ for non-negative integers?

Category 1 x_1	Category 2 x_2	Category 3 x_3	Result of Selection
xxxxxxx		xxxx	7 + 0 + 4
xx	xxxxxx	xxx	2 + 6 + 3
	xxxxxxxxxx	x	0 + 10 + 1

Arranging 11 x and 2 | in the 13 positions. Choose where to put | and put the x in the remaining positions: $C(13,2) = 13!/(2!11!) = 78$

Combinations with Repetitions Allowed

How many solutions to the equation $x_1 + x_2 + x_3 = 11$ for positive integers?

Category 1 x_1	Category 2 x_2	Category 3 x_3	Result of Selection
xxxxx x	 x	xxx x	6 + 1 + 4
x x	xxxxx x	xx x	2 + 6 + 3
 x	xxxxxxxx x	 x	1 + 9 + 1

Put one x into each category to ensure it is non-zero now the remaining 8 x and 2 | can be arranged in $C(10,2) = 45$ ways

Summary

1. n different objects in **n** spaces, order important — $n!$
2. n different objects in **r** spaces, order important — $P(n,r) = n!/(n-r)!$
3. n different objects in **r** spaces, order **not** important — $C(n,r) = \binom{n}{r} = n!/[(n-r)!r!] = P(n,r)/r!$

Multiplication Rule: object consists of a **sequence** of independent choices (AND)

Addition Rule: object consists of a **collection** of independent choices (OR)

Binomial Coefficients

$(a + b)^4 = (a + b)(a + b)(a + b)(a + b)$

$$= \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4$$

Binomial Theorem: Let *x* and *y* be variables, and let *n* be any nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Binomial Coefficients

Binomial Theorem: Let *x* and *y* be variables, and let *n* be any nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

What is the coefficient of a^8b^9 in the expansion of $(3a+2b)^{17}$?

What is n? 17

What is j? 9

What is x? 3a

What is y? 2b

$$\binom{17}{9}(3a)^8(2b)^9 = \binom{17}{9}3^8 2^9 a^8 b^9$$

CS202 Fall 2011
Lecture 6 – 9/8

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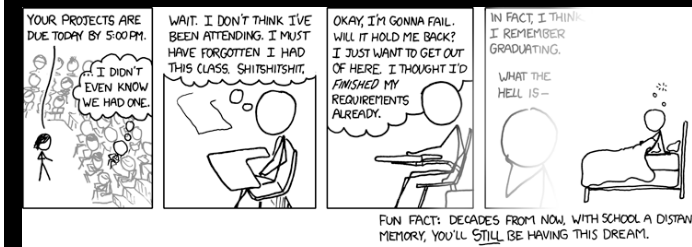
Combinatorics: Counting



xkcd comic: http://xkcd.com/557/

Binomials: Application Example

- How many PINs (of letters and numbers) of length 4 contain 3 letters and 1 number?
- For a query "zebra", a search engine returns 8,324,154 results about the animal and 46,537 results related to software and technology. If the results are unranked and repetitions are allowed, how many Top 10 lists of the results contain 7 animal and 3 technology results?

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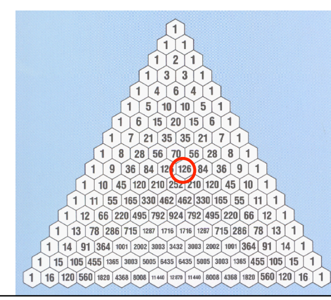
Binomial Coefficients

$(a+b)^4 = (a+b)(a+b)(a+b)(a+b)$

$$= \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4$$



A. C(10,6)
B. C(9,4)
C. C(9,5)
D. C(8,4) + C(8,5)
E. No clue

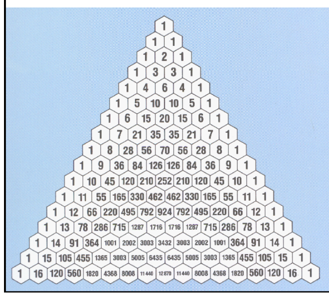
Binomial Coefficients

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Sum each row of Pascal's Triangle:

Powers of 2

Two proofs that $\sum_{j=0}^{n} \binom{n}{j} = 2^n$



Suppose you have a set of size n. How many subsets does it have? 2^n

How many subsets of size 0 does it have? C(n,0)

How many subsets of size 1 does it have? C(n,1)

How many subsets of size 2 does it have? C(n,2)

Count all subsets in this way, and we have the result!

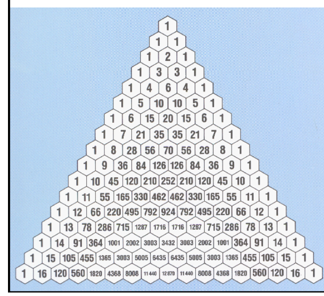
Binomial Coefficients

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Sum each row of Pascal's Triangle:

Powers of 2

Two proofs that $\sum_{j=0}^{n} \binom{n}{j} = 2^n$



Let x=1 and y=1 in Binomial Theorem.

Done

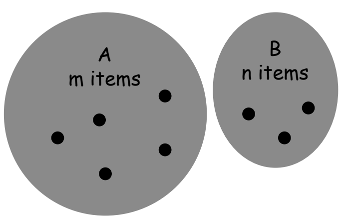
$$\sum_{j=0}^{n} \binom{n}{j} 1^{n-j} 1^j = (1+1)^n$$

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n$$

Vandermonde's Identity

Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{j=0}^{r} \binom{m}{r-j} \binom{n}{j}$$



To choose r items, take some from A and some from B. All possible ways of doing this gives the result.