University of Illinois at Chicago
Department of Computer Science

Final Examination

CS 151 Mathematical Foundations of Computer Science
Fall 2012

Thursday, October 18, 2012

Name:
Email:

- **Print your name and email, neatly in the space provided above; print your name at the upper right corner of *every page*. Please print legibly.**
- **This is a *closed book* exam. You are permitted to use only your brain. *Nothing else is permitted.***
- ***Show your work!* You will not get partial credit if we cannot figure out how you arrived at your answer.**
- **Write your answers in the space provided for the corresponding problem. Let us know if you need more paper.**
- **If any question is unclear, ask us for clarification.**
- **GOOD LUCK!**

Q	Pts	Score	Grader
Logic	55		
Induction1	20		
Induction2	25		
Total	100		

1. **Logic.**

(a) (4 points) Write the truth table for the compound proposition $(p \to \neg q) \wedge (\neg r \oplus q)$
Answer:

p	q	r	$p \leftarrow \neg q$	$\neg r \oplus q$	$(p \to \neg q) \wedge (\neg r \oplus q)$
T	T	T	F	T	F
T	T	F	F	F	F
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	T	T	T

(b) (4 points) Give a logically equivalent proposition to the one above
Answer:
$(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$
Or,
$(\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee \neg r)$

(c) (4 points) Are the following system specifications consistent? "Whenever the system software is being upgraded, users cannot access the system. If users can access the system, then they can save new files. If users cannot save new files, then the system software is not being upgraded."
Answer:
The premise is:

$$upgrade \to \neg access \quad (1)$$
$$access \to savefile \quad (2)$$
$$\neg savefile \to \neg upgrade \quad (3)$$

The staments does not contradicts each other. So they are consistent.

(d) (4 points each) Let $P(x)$:=“Student x gets an A on the midterm”, $Q(x,y)$:= “Student x does exercise y in the book”, $R(x)$:= “Student x gets an A in this class”. Write the following propositions using P, Q, and R, quantifiers and logical connectives.

i. There are students who get an A in this course but do not do every exercise in the book.
Answer:
$\exists x(R(x) \wedge \neg \forall y Q(x,y))$

ii. All students who get an A on the midterm and do every exercise in the book get an A in this class.
Answer:
$\forall x(P(x) \wedge \forall y Q(x,y) \rightarrow R(x)$

iii. There are students who do not get an A on the midterm but get an A in this class.
Answer:
$\exists x \neg P(x) \wedge R(x)$

iv. To get an A in this class it is necessary for student Fluffy to get an A on the midterm.
Answer:
$R(\text{Fluffy}) \rightarrow P(\text{Fluffy})$

v. Write the converse, contrapositive, and the inverse of the statement (iii) above.
Answer:
converse:
$P(\text{Fluffy}) \rightarrow R(\text{Fluffy})$

contrapositive:
$\neg P(\text{Fluffy}) \rightarrow \neg R(\text{Fluffy})$

inverse:
$\neg R(\text{Fluffy}) \rightarrow \neg P(\text{Fluffy})$

vi. Translate the following statement into English:

$$\forall x \in \text{Students } \exists y \in \text{Exercises } Q(x, y)$$

Answer:
All students do some exercises.

vii. Translate the following statement into English:

$$\forall x \in \text{Students } \forall y \in \text{Exercises } (Q(x, y) \rightarrow R(x))$$

Answer:
(For all students,) if a student does all exercises, then s/he will get an A in class.

viii. Write the negation of the statement above and translate it into English.
Answer:

$$\exists x \in \text{Students } \exists y \in \text{Exercises } (Q(x, y) \wedge \neg R(x))$$

So, there are students who did some exercises but did not get A in class.

(e) (4 points) Determine whether the following argument is correct and explain why. All students who get an A in this class do well in the undergraduate program and get a great job. Puffy got a great job. Puffy was a student in this class. Therefore, Puffy got an A on tin this class.

Answer:

Premise is: $\forall x \in$ Students

$$getA(x) \rightarrow doWell \wedge getJob(x)$$
$$\implies getA(x) \rightarrow getJob(x)$$

Claim is:

$$getJob(\text{Puffy}) \rightarrow getA(\text{Puffy})$$

which is a converse of the premise with instance Puffy. And we know that converse of a implication is not necessarily true. So, the argument is incorrect.

(f) (7 points) Prove that if n is an integer an $3n + 2$ is even, then n is even using a proof by contradiction.

Answer:

Suppose, k in an integer and $n = 2k + 1$, so n is an odd number.

Then, $3n + 2 = 3(2k + 1) + 2 = 6k + 3 + 2 = 6k + 5$

Since $6k$ is always even for any integer k, $6k + 5$ is odd, therefore $3n + 2$ is odd. Which is a contradiction. So our assumption of n being odd cannot be correct. Hence n is even.

2. **Algebraic Induction (20 points).**

In the following steps you will prove by induction the following claim:

"For all positive integers n $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

(a) (2 points) We would like to have the claim in the form $\forall n \geq n_0\ P(n)$. What should be the constant n_0 and the proposition $P(n)$?

Answer:

$n_0 = 1$

$\forall n \geq 1, P(n) :=$ $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

(b) (2 points) What is the statement $P(n_0)$ (for the n_0 you defined above)?

Answer:

$P(1) :=$ $\sum_{i=1}^{1} i^3 = \left(\frac{1(1+1)}{2}\right)^2$

(c) (3 points) Show that $P(n_0)$ is true (for the n_0 you defined above), thus completing the Base Case of the proof.

Answer:

$\sum_{i=1}^{1} i^3 = \left(\frac{1(1+1)}{2}\right)^2$

$\implies 1^3 = \left(\frac{2}{2}\right)^2$

$\implies 1 = 1$

(d) (3 points) What is the Inductive Hypothesis?
Answer:
$P(n-1) := \quad \sum_{i=1}^{n-1} i^3 = \left(\frac{(n-1)n}{2}\right)^2$
Or,
$P(n) := \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

(e) (3 points) Write the statement of the Inductive Step.
Answer:
$P(n-1) \rightarrow P(n)$
Or,
$P(n) \rightarrow P(n+1)$

(f) (5 points) Complete the proof of the Inductive Step, indicating where you use the Inductive Hypothesis.
Answer:
Let inductive hypothesis was $P(n-1)$, So let us prove $P(n)$
$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{n-1} i^3 + n^3 = \left(\frac{(n-1)n}{2}\right)^2 + n^3$ [by inductive hypothesis]
$= \frac{n^4-2n^3+n^2}{4} + n^3 = \frac{n^4+2n^3+n^2}{4} = \frac{(n^2+2n+1)n^2}{4} = \left(\frac{n(n+1)}{2}\right)^2$

(g) (2 points) Explain why these steps show that this formula is true whenever n is a positive integer.
Answer:
Since the basis step and inductive step hold, by the priciple of mathematical induction, $P(n)$ is true for all $n \geq 1$.

3. **Structural Induction (25 points).**
We will define a special set S of strings. A string of length 1 is "a". A string of length 2 is either "aa", "ba", or "ca". If s_1 and s_2 are strings then $s_1 s_2 a$ (the concatenation (gluing together) of strings s_1, s_2 and "a") is also a string. Let $P(n)$ be the statement "there are more a's than other letters in strings of length n". We will prove by strong induction (on the length of a string) the claim that $P(n)$ is true for all $n \geq 3$.

(a) (2 points) What is the statement $P(3)$?
Answer:
There are more a's than other letters in strings of length 3.

(b) (2 points) Write all the strings of length 3 in S
Answer:
aaa.

(c) (3 points) Explain why $P(3)$ is true, thus completing the Base Case of the proof. Explain why $n = 3$ must be the base case.
Answer:
The only string of length 3 is aaa which is obviously more a's than other letters.

(d) (4 points) What is the Inductive Hypothesis?
Answer:
$\forall k \geq 3,\ k < n,\ P(k)$

(e) (3 points) Write the statement of the Inductive Step.
Answer:
$\forall k \geq 3,\ k < n,\ P(k) \rightarrow P(n)$

(f) (7 points) Complete the proof of the Inductive Step, indicating where you use the Inductive Hypothesis.

Answer:

Let s be a string of length n. By definition, we can write $s = s_1 s_2 a$. By inductive hypothesis both s_1 and s_2 contains more a than other letters. And the last a adds up to the number of a one more. So, total number of a's in $s_1 s_2 a$ is more than any other letters. Which proves inductive step.

(g) (2 points) Explain why these steps show that this formula is true whenever n is a positive integer.

Answer:

Since both basis case and inductive inductive step are proved, by the principle of strong induction, the formula is true.