

# Acceleration of the Convergence of Series Containing Mathieu Functions Using Shanks Transformation

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**Abstract**—A modification of the standard application of Shanks transformation is shown to improve the convergence rate in certain cases where the straightforward application of Shanks transformation fails. Here, the straightforward application of Shanks transformation to a well known series expansion containing Mathieu functions failed to improve the convergence rate. However, convergence was achieved by a new method of applying Shanks transformation. This new method requires analysis of the behavior of the series terms to determine the cause of the slow or failing convergence. Then Shanks transformation was applied only to the slowly convergent part of the series. This work is important because with this new method convergence may be achieved in cases where the standard application of Shanks transformation fails to improve the converge rate.

**Index Terms**—Boundary value problems, convergence rate, Mathieu function, series acceleration, Shanks transform.

## I. INTRODUCTION

**S**ERIES acceleration techniques are used in many applications. There is no general acceleration method that works in all cases, but rather different methods are successful depending upon the particular case [1]. Among various acceleration techniques, the method of Shanks transformation has proven to be helpful in accelerating the convergence of a series by eliminating the most pronounced transient behavior of the sequence of partial sums. An attractive feature of Shanks transformation is that it is easy to apply. Unfortunately, there are situations where its straightforward application does not increase the convergence rate. Here, one such situation, a series expansion of a well-known function, is examined. This work is novel because it shows how to modify the application of Shanks transformation so that the convergence rate may be accelerated for situations where its direct application fails. This method is not expected to guarantee convergence in all cases, but it has been successfully applied to the series expansions computed in [2] where the direct application of Shanks transformation failed and is important in applications such as the computation of solution of boundary value problems.

## II. CASE STUDY: THE EXPANSION OF A CYLINDRICAL WAVE IN A SERIES OF MATHIEU FUNCTIONS

In some electromagnetic problems, the geometry is best described using elliptic coordinates. With this coordinate system,

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Mathieu functions constitute the natural choice for the basis functions. This is the case, for example, of a line source that radiates toward either an elliptic cylinder [3] or a plane with an infinite slot containing a semielliptical cavity [4]. In these problems, the solutions for the scattered fields may be expressed as series expansions containing Mathieu functions because it is easy to apply the boundary conditions along the coordinate surfaces and then determine the unknown fields by mode matching. The resulting series are usually absolutely convergent but in some cases their convergence is slow, and, therefore, the use of an acceleration technique is required. Let us now consider an example of such a behavior.

Assuming a plane geometry, suppressing the time-dependent factor  $\exp(j\omega t)$ , a cylindrical wave coming from a line source located at  $(x_0, y_0)$ , with wavevector  $k$ , is written

$$H_0^{(2)}(kR) \quad (1)$$

where  $H_0^{(2)}$  is the Hankel function of second kind of order zero and  $R$  is the distance between the observation point  $(x, y)$  and  $(x_0, y_0)$ , i.e.,

$$R = \sqrt{(x - x_0)^2 + (y - y_0)^2}. \quad (2)$$

Consider then an elliptic coordinate system with foci located at  $(\pm d/2, 0)$ , i.e., with focal distance  $d$ . The relationship between elliptic coordinates  $(u, v)$  and Cartesian coordinates  $(x, y)$  is

$$x = \frac{d}{2} \cosh u \cos v \quad (3)$$

$$y = \frac{d}{2} \sinh u \sin v. \quad (4)$$

The Hankel function (1) line is expressed as a series expansion of Mathieu functions as

$$\begin{aligned} H_0^{(2)}(kR) = & 4 \sum_{m=0}^{\infty} \frac{1}{N_m^{(e)}} \text{Re}_m^{(1)}(s, u_{<}) \text{Re}_m^{(4)}(s, u_{>}) \text{Se}_m(s, v_0) \\ & \times \text{Se}_m(s, v) + \frac{1}{N_m^{(o)}} \text{Ro}_m^{(1)}(s, u_{<}) \text{Ro}_m^{(4)}(s, u_{>}) \\ & \times \text{So}_m(s, v_0) \text{So}_m(s, v) \end{aligned} \quad (5)$$

where  $(u_0, v_0)$  are the elliptic coordinates corresponding to  $(x_0, y_0)$ ,  $(u, v)$  those corresponding to  $(x, y)$  and where  $u_{<}$  ( $u_{>}$ ) is the smaller (larger) between  $u$  and  $u_0$ . The parameter  $s$  is given by  $(kd)^2/16$ . Series (5) contains the angular Mathieu functions  $\text{Se}_m, \text{So}_m$ , which are periodic functions of  $v$ , with even parity for  $\text{Se}_m$  and odd for  $\text{So}_m$ . The functions  $\text{Re}_m^{(1)}, \text{Re}_m^{(4)}, \text{Ro}_m^{(1)}$ , and  $\text{Ro}_m^{(4)}$  are the radial Mathieu functions that have even parity for  $\text{Re}_m$  and odd parity for  $\text{Ro}_m$ . Series (5) requires that the

TABLE I  
CASE STUDY PARAMETERS

parameter	value
source location	$(x_0 = 0, y_0 = 0)$
field location	$(x = 1, y = 1)$
distance	$R = \sqrt{2}$
focal distance	$d=2$
wavevector	$k=2$
Mathieu function parameter	$s = 1$
reference value	
$H_0^{(2)}(kR)$	$= -0.196548095270468 - j0.428287398117323$

Mathieu functions be normalized according to Stratton's definition, which is reported in [5] and [6], together with the definitions of the normalization coefficients  $N_m^{(e)}$  and  $N_m^{(o)}$ . Further details about the computation of the Mathieu functions are given in the Appendix.

In some situations, series (5) may become slowly convergent. This case study analyzes one such situation for the values of the parameters that are given in Table I, which, in particular, gives the sum of series (5). Table II shows the values of some of the first 19 terms of the series together with the corresponding partial sums. It is clear from Table II that when  $n = 8$  the real part of  $S_8$  has converged to the reference value, while the imaginary part has not. When  $n = 18$ , the imaginary part of  $S_n$  moves further away from the reference value. Therefore, the reason for the slow convergence of (5) is due to the behavior of the imaginary part of each term of the expansion. In order to increase the convergence rate of series (5), one may apply Shanks transformation [7], [8] to this series. Shanks transformation is applied as follows. Consider a series of the form

$$S = \sum_{k=0}^{\infty} a_k \quad (6)$$

with partial sums given by

$$S_n = \sum_{k=0}^n a_k. \quad (7)$$

Shanks transformation consists in creating a table with the following entries:

$$e_{s+1} = e_{s-1}(S_{n+1}) + \frac{1}{e_s(S_{n+1}) - e_s(S_n)} \quad (8)$$

where

$$e_0(S_n) = S_n \quad (9)$$

$$e_1(S_n) = \frac{1}{e_0(S_{n+1}) - e_0(S_n)}. \quad (10)$$

The even-order terms  $e_{2r}(S_n)$  are Shanks transforms of order  $r$  approximating  $S$ , while the odd-order term  $e_{2r+1}(S_n)$  are merely intermediate quantities. The algorithm is stopped when the following convergence criterion is satisfied:

$$\left| \frac{e_{2r+2}(S_{n-1}) - e_{2r}(S_n)}{e_{2r+2}(S_n)} \right| \leq \varepsilon_c. \quad (11)$$

One may also check that three successive values of  $e_{2r+2}(S_{n-1})$ ,  $e_{2r+4}(S_{n-2})$ ,  $e_{2r+6}(S_{n-3})$  satisfy (11) to ensure that the process is not stopped prematurely.

When Shanks transformation is applied to the terms of Table II, the result is shown in Table III. Column  $e_0(S_n)$  simply represents the partial sums, according to (9), and does not add anything new. However, the intermediate quantities  $e_1(S_n)$  contain diverging terms since the denominator of (10) is numerically zero. This happens because the imaginary part of successive terms  $a_n$  in Table II undergo large variations of many orders of magnitude, which causes successive values of  $\text{Imag}(S_n)$  to be numerically identical. Additionally, when  $n > 8$ ,  $\text{Imag}(a_n)$  can still have large values, on the order of  $10^1$  when  $n = 18$ . In contrast, the exam of the real part of column  $a_n$  of Table II shows that when  $n > 8$ , successive terms of  $a_n$  introduce variations that make  $\text{Real}(S_n)$  change by no more than  $10^{-14}$  in absolute value. Because the slow convergence appears related to the behavior of  $\text{Imag}(a_n)$ , this suggests that Shanks transformation can be applied in the following way. Series (5) can be split into its real and imaginary parts as

$$H_0^{(2)}(kR) = \sum_{n=0}^{\infty} a_n = \underbrace{\sum_{n=0}^{\infty} \text{Real}(a_n)}_{S^{\text{Real}}} + j \underbrace{\sum_{n=0}^{\infty} \text{Imag}(a_n)}_{S^{\text{Imag}}} \quad (12)$$

so that  $\text{real}S \approx \text{real}(S_n)$ . Then Shanks transformation is applied only to the imaginary part of column  $a_n$  of Table II. When this is accomplished, one obtains the results shown in Table IV.

One observes that the same problem previously encountered with the entries  $e_1(S_n)$  of Table III is repeated here. In fact, some terms of the intermediate columns of odd order, such as  $e_1(S_n)$ ,  $e_3(S_n)$ , diverge for the same reason encountered before. In order to eliminate the numerical zero at the denominator of (8), one still needs to operate only on the imaginary part of  $a_n$ . Because the sequence of  $\text{Imag}(a_n)$  alternates values which are large and small in absolute value, the sequence  $\text{Imag}(a_n)$  may be split in even- and odd-order terms:

$$S^{\text{Imag}} = \underbrace{\sum_{k=0}^{\infty} \text{Imag}(a_{2k})}_{S^{\text{Imag}}_{\text{Even}}} + \underbrace{\sum_{k=0}^{\infty} \text{Imag}(a_{2k+1})}_{S^{\text{Imag}}_{\text{Odd}}}. \quad (13)$$

When one accomplishes this split, the two series at the right hand side of (13) do not show the alternating behavior of large and small terms, in absolute value, that caused the lack of convergence. After having preprocessed series (5) according to (12) and (13), the corresponding results are collected in Tables V and VI. One can see that now convergence is achieved with  $\varepsilon_c = 10^{-4}$ . Therefore, the sum of series (5) is approximated with

$$\begin{aligned} S &\approx \text{Real}(S_8) + j \left( e_4(S^{\text{Imag}}_{\text{Even}}) + e_4(S^{\text{Imag}}_{\text{Odd}}) \right) \\ &= -0.1965480953 - j0.4282876370. \end{aligned} \quad (14)$$

Finally, one may ask why the imaginary part of series (5) contains such a strong oscillating behavior. The answer stems from the definition of Mathieu functions. All Mathieu functions

TABLE II  
PARTIAL SUM FOR THE CYLINDRICAL WAVE EXPANSION

n	$a_n$		$S_n$	
0	-0.1808027322E+00	-j0.4130154378E+00	-0.1808027322E+00	-j0.4130154378E+00
1	0.3063580180E-16	-j0.3128493776E-16	-0.1808027322E+00	-j0.4130154378E+00
2	-0.1459183878E-01	-j0.5310398913E-02	-0.1953945709E+00	-j0.4183258367E+00
...	...	...	...	...
8	0.9093548455E-10	+j0.2776398838E-04	<b>-0.1965480953E+00</b>	<b>-j0.4282908838E+00</b>
...	...	...	...	...
18	-0.4827771116E-21	-j0.9448007615E+01	-0.1965480953E+00	-j0.9871304048E+01

TABLE III  
SHANKS TRANSFORMATION APPLIED TO THE CYLINDRICAL WAVE EXPANSION

n	$e_0(S_n)$		$e_1(S_n)$	
0	-0.1808027322E+00	-j0.4130154378E+00	0.7205759404E+16	+j0.1441151881E+17
1	-0.1808027322E+00	-j0.4130154378E+00	-0.6051639001E+02	+j0.2202369260E+02
2	-0.1953945709E+00	-j0.4183258367E+00	Infinity	Infinity
3	-0.1953945709E+00	-j0.4183258367E+00	-0.1218434182E+02	+j0.1020717332E+03
4	-0.1965476163E+00	-j0.4279852289E+00	Infinity	Infinity

TABLE IV  
SHANKS TRANSFORMATION APPLIED TO THE IMAGINARY PART OF THE CYLINDRICAL WAVE EXPANSION

n	$e_0(S_n^{Imag})$	$e_1(S_n^{Imag})$	$e_2(S_n^{Imag})$	$e_3(S_n^{Imag})$	$e_4(S_n^{Imag})$
0	-0.4130154378E+00	-0.1801439851E+17	-0.4130154378E+00	-0.3766195408E+03	-0.4183258367E+00
1	-0.4130154378E+00	-0.1883097704E+03	-0.4183258367E+00	Infinity	-0.4183258367E+00
2	-0.4183258367E+00	Infinity	-0.4183258367E+00	-0.2070523655E+03	-0.4279852289E+00
3	-0.4183258367E+00	-0.1035261828E+03	-0.4279852289E+00	Infinity	-0.4279852289E+00
4	-0.4279852289E+00	Infinity	-0.4279852289E+00	-0.5998460790E+04	-0.4283186478E+00

TABLE V  
SHANKS TRANSFORMATION APPLIED TO THE EVEN TERMS OF THE IMAGINARY PART OF THE CYLINDRICAL WAVE EXPANSION

n	$e_0(S_n^{Imag,even})$	$e_1(S_n^{Imag,even})$	$e_2(S_n^{Imag,even})$	$e_3(S_n^{Imag,even})$	$e_4(S_n^{Imag,even})$
0	-0.4130154378E+00	-0.1883097704E+03	-0.4065311011E+00	-0.1493988641E+03	-0.4282885182E+00
1	-0.4183258367E+00	-0.1035261828E+03	-0.4283305680E+00	0.2363189017E+05	-0.4282872082E+00
2	-0.4279852289E+00	-0.2999230395E+04	-0.4282930180E+00	0.1957550170E+06	-0.4282873640E+00
3	-0.4283186478E+00	0.3601787993E+05	-0.4282867577E+00	-0.1453489112E+07	-0.4282867493E+00
4	-0.4282908838E+00	0.2783787984E+06	-0.4282873351E+00	0.2536358669E+06	-0.4282876370E+00

TABLE VI  
SHANKS TRANSFORMATION APPLIED TO THE ODD TERMS OF THE IMAGINARY PART OF THE CYLINDRICAL WAVE EXPANSION

n	$e_0(S_n^{Imag,odd})$	$e_1(S_n^{Imag,odd})$	$e_2(S_n^{Imag,odd})$	$e_3(S_n^{Imag,odd})$	$e_4(S_n^{Imag,odd})$
0	-0.3128493776E-16	0.1264417490E+18	-0.2357656420E-16	-0.1431260027E+20	-0.2365071197E-16
1	-0.2337615756E-16	-0.4863412990E+19	-0.2368239341E-16	0.1725161962E+20	-0.2364811516E-16
2	-0.2358177448E-16	-0.1480190142E+20	-0.2365119559E-16	0.3418818229E+21	-0.2364850090E-16
3	-0.2364933337E-16	-0.5517970591E+21	-0.2365007662E-16	0.9765155147E+21	-0.2364964339E-16
4	-0.2365114563E-16	0.3836422435E+21	-0.2364838992E-16	0.1787306331E+21	-0.2364840165E-16

$Re_m^{(1)}$ ,  $Ro_m^{(1)}$  are defined as series of Bessel functions  $J$  that are well behaved. The Mathieu functions  $Re_m^{(4)}$  and  $Ro_m^{(4)}$  are defined as:

$$Re_m^{(4)} = Re_m^{(1)} - jRe_m^{(2)} \quad (15)$$

$$Ro_m^{(4)} = Ro_m^{(1)} - jRo_m^{(2)} \quad (16)$$

where  $Re_m^{(2)}$ ,  $Ro_m^{(2)}$  contain Bessel functions  $Y$  which are singular at the origin. The slow convergence of the example consid-

ered here is caused by the relatively small value for the argument of the functions  $Re_m^{(2)}$  and  $Ro_m^{(2)}$ .

### III. CONCLUSION

A slowly convergent series was considered for which the direct application of Shanks transformation did not improve its convergence. However, by examining the behavior of the terms of the series, that part of the series that converged slowly was isolated. Then, Shanks transformation was applied only

to the slowly convergent part and convergence was achieved. This modification to the application of Shanks transformation may achieve convergence where its straightforward application fails. There are many other cases involving series expansions containing Mathieu functions that behave similarly to the one discussed here, for example, those given in [2].

#### APPENDIX

The author has used, in part, some Fortran subroutines contained in [9] to compute the necessary Mathieu functions. The notation used in [9] is the one of Goldstein-Ince, which differs from Stratton's notation [5], [6] used here. Therefore, some clarifications are necessary for computational purposes. The subroutines given in [9] compute the even angular functions  $ce_m$ , the odd angular functions  $se_m$ , the even radial functions  $Mc_m^{(1,2)}$  and the odd radial functions  $Ms_m^{(j)}$ . The necessary relationships are

$$Se_m(s, v) = \frac{ce_m(s, v)}{ce_m(s, 0)} \quad (17)$$

$$So_m(s, v) = \frac{se_m(s, v)}{\frac{d}{dv} se_m(s, v)|_{v=0}} \quad (18)$$

$$Re_m^{(1,2)}(s, u) = \sqrt{\frac{\pi}{2}} Mc_m^{(1,2)}(s, u) \quad (19)$$

$$Ro_m^{(1,2)}(s, u) = \sqrt{\frac{\pi}{2}} Ms_m^{(1,2)}(s, u). \quad (20)$$

#### REFERENCES

- [1] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*. New York: Springer-Verlag, 1999.
- [2] D. Erricolo, M. D. Lockard, C. M. Butler, and P. L. E. Uslenghi, "Numerical analysis of penetration, radiation and scattering for a slotted semielliptical channel filled with isorefractive material," *IEEE Trans. Antennas Propagat.*, submitted for publication.
- [3] J. J. Bowman, T. B. A. Senior, and P. L. E. Uslenghi, *Electromagnetic and Acoustic Scattering by Simple Shapes*. New York: Hemisphere, 1971.
- [4] P. L. E. Uslenghi, "Exact penetration, radiation and scattering for a slotted channel filled with isorefractive material," *IEEE Trans. Antennas Propagat.*, to be published.
- [5] J. A. Stratton, *Electromagnetic Theory*. New York: McGraw-Hill, 1941.
- [6] National Bureau of Standards, *Tables Relating to Mathieu Functions*. New York: Columbia Univ. Press, 1951.
- [7] S. Singh, W. F. Richards, J. R. Zinecker, and D. R. Wilton, "Accelerating the convergence of series representing the free space periodic green's function," *IEEE Trans. Antennas Propagat.*, vol. 38, pp. 1958–1962, Dec. 1990.
- [8] D. Shanks, "Non-linear transformation of divergent and slowly convergent series," *J. Math. Phys.*, vol. 34, pp. 1–42, 1955.
- [9] S. Zhang and J. M. Jin, *Computation of Special Functions*. New York: Wiley, 1996.